$$\theta=\begin{bmatrix} V_3 \\ -I_4 \end{bmatrix}=\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}\begin{bmatrix} -I_3 \\ V_4 \end{bmatrix}=X\phi. \tag{13}$$

From (12) and (13) we can solve I_1 and V_2 in terms of V_1 and I_2 and the result is given by

$$\begin{bmatrix} I_1 \\ V_2 \end{bmatrix}=\left\{\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}+\begin{bmatrix} H_{13} & -H_{14} \\ H_{23} & -H_{24} \end{bmatrix}\cdot\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}^{-1}\begin{bmatrix} -H_{31} & -H_{32} \\ H_{41} & H_{42} \end{bmatrix}\right\}\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} \tag{14}$$

where

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}=\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}^{-1}+\begin{bmatrix} H_{33} & -H_{34} \\ -H_{43} & H_{44} \end{bmatrix}, \tag{15}$$

provided that the various inverses exist. The coefficient matrix of (14) defines the general hybrid matrix of the n-port network as shown in Fig. 2. Comparing this coefficient matrix with (5), we can make the following identifications:

$$D=\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix},\quad C=\begin{bmatrix} H_{13} & -H_{14} \\ H_{23} & -H_{24} \end{bmatrix}, \tag{16a}$$

$$A=-\begin{bmatrix} H_{33} & -H_{34} \\ -H_{43} & H_{44} \end{bmatrix},\quad B=\begin{bmatrix} -H_{31} & -H_{32} \\ H_{41} & H_{42} \end{bmatrix}. \tag{16b}$$

From (6) the return difference matrix is found to be

$$F(X)=1_p-AX=\tilde{H}^{-1}(\tilde{H}+X), \tag{17}$$

where

$$\tilde{H}=\begin{bmatrix} H_{33} & -H_{34} \\ -H_{43} & H_{44} \end{bmatrix}^{-1}. \tag{18}$$

Also, as can be seen from (12), by setting $V_1=0$ and $I_2=0$ we have

$$\begin{bmatrix} V_3 \\ I_4 \end{bmatrix}=\begin{bmatrix} H_{33} & H_{34} \\ H_{43} & H_{44} \end{bmatrix}^{-1}\begin{bmatrix} I_3 \\ V_4 \end{bmatrix}\Bigg|_{V_1=0 \text{ and } I_2=0}. \tag{19}$$

The coefficient matrix of (19) is precisely the hybrid matrix looking into the p ports of N_1 when all the independent current sources are open-circuited and all the independent voltage sources are short-circuited.

In the particular situation where X denotes the admittance matrix of the p-port network N_2, the return difference matrix becomes

$$F(X)=Y^{-1}(Y+X) \tag{20}$$

where Y is the admittance matrix facing the p-port network N_2 in Fig. 2 with $V_1=0$ and $I_2=0$. The matrix $Y+X$ represents the total admittance matrix looking into the junctions of the p-port network N_2 and the multiport network N_1. This is therefore a direct generalization of the scalar return difference with respect to a one-port admittance x, which is equal to the ratio of the total admittance looking into the node pair where x is connected to the admittance y that x faces, as indicated in (9). A similar interpretation can be made if X denotes the impedance matrix of the p-port network N_2. In this case, the return difference matrix becomes

$$F(X)=Z^{-1}(Z+X) \tag{21}$$

where Z is the impedance matrix facing the p-port network N_2 in Fig. 2 with $V_1=0$ and $I_2=0$.

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Use of Time-Delay Actions in the Controller Design

IL HONG SUH AND ZEUNGNAM BIEN

***Abstract*—It is shown that time-delay action can be used in the design of controllers having quick settling-time property. Specifically, when the feedback compensator is implemented by means of proportional minus delay action, the output performance of the closed-loop system can be better in the sense of ITAE index than that of the system which employs a conventional proportional-plus-derivative action. For actual design, a rule of thumb for determining the parameters of proportional minus delay feedback compensator is suggested.**

I. Introduction

The time-delay, when appropriately incorporated in the controller, may improve the performance of the controlled system. Several reports are available on positive uses of the time-delay actions in the controller design. Tallman and Smith [1] and Rubin [2] have shown that a second-order lightly damped oscillatory system can be controlled to produce a deadbeat nonoscillatory response by a controller with time-delay. Choksy [3] briefly explained that an unstable linear time-invariant system can be made stable by introducing a delayed feedback. For the control of processes with inherent time-delays, Smith [4] suggested that a controller with minor-loop feedback, the so-called Smith's linear predictor, be used. This scheme utilizes the action of the time-delay, the length of which is equal to the plant itself. Gilchrist [5] has used time-delay actions to construct the exact state vectors from the output. In [6], we suggested a new kind of controller utilizing proportional minus delay actions, and showed that the output performance of a controlled process with the suggested controller is similar to those of controlled processes with proportional plus derivative (PD) controller, but is superior in responding to noise-like disturbances.

In this note, it is reported that the time-delay action may be used in the design of feedback controllers with a fast settling-time property. When the proportional minus delay action is appropriately incorporated in the feedback path for compensation, it is shown that the performance of the resulting control system is definitely better than those systems employing the conventional, optimally adjusted, proportional-plus derivative scheme. In our investigation, a typical second-order system is taken as a controlled plant and the step responses for the conventional and the suggested schemes are compared with regard to the well-known ITAE-index. Further, for practical use, a rule of thumb for near-optimally setting the proportional minus delay controller in an arbitrary second-order system is provided.

In the sequel, the abbreviations PD, PMD, and ITAE will stand for the proportional plus derivative, the proportional minus delay and the integral of the time multiplied by the absolute error, respectively.

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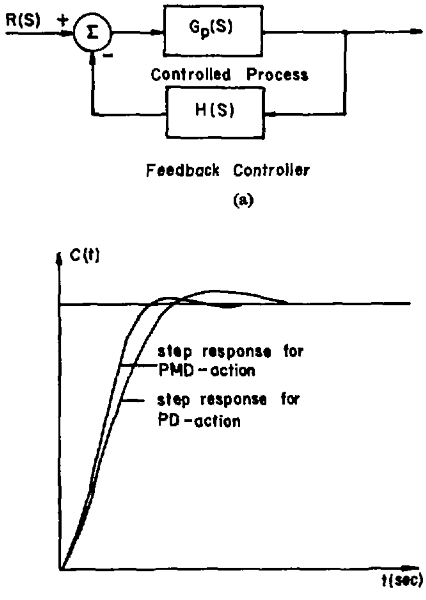

Fig. 1. (a) System block diagram and (b) step input responses.

II. Design of Feedback Compensator via PMD-Action

A. Background

Consider the second-order plant whose dynamics is given by the following transfer-function:

$$G_{p(s)}=\frac{\omega_0^2}{s(s+a)}. \tag{1}$$

When the conventional PD-action compensator of the form $H(s)=1+ks$ is used in the feedback path as shown in Fig. 1(a), the overall transfer-function becomes

$$\frac{C(s)}{R(s)}=\frac{\omega_0^2}{S^2+2\xi\omega_0 s+\omega_0^2} \tag{2}$$

where $\xi=(a+k\omega_0^2)/2\omega_0$. It is well known [7] that when k is adjusted in such a way that the system damping ratio ξ is 0.7, then the ITAE index is minimum. It is also observed that, for the second-order system (1), the PD-action $H(s)=1+ks$ is a state-feedback action, and in modern control theory [11], the system with $\xi=0.7$ is an optimal control system in the sense that the state feedback gain was determined to minimize a quadratic cost of the type

$$J=\lim_{\rho\to 0} 1/2\int_0^\infty (\rho\|Gx\|^2+u^2)dt.$$

It is found, however, that when the feedback compensator is implemented by means of the PMD-action, the performance of the closed-loop system can be better than that of the system which employs the conventional PD-action. Specifically, when the feedback compensator $H(s)$ is designed to be

$$H(s)=1+k\left(\frac{1-e^{-hs}}{h}\right)=\left(1+\frac{k}{h}\right)-\frac{k}{h}e^{-sh} \tag{3}$$

and when the parameters h and k are chosen in a suitable manner, the step response of the resulting system reveals a quicker settling-time property. In terms of the ITAE index, the proposed control system attains considerably less index value than the conventional one. This is specifically illustrated in the following example.

Example 1: Let the dynamics of the controlled system $G_p(s)$ be given by

$$G_p(s)=\frac{10^2}{s(s+2)} \tag{4}$$

and suppose the unit step input $r(t)$ is applied to the initially relaxed system. When a conventional PD-action is used in the feedback compensator, the optimum design of $H(s)$ is easily computed as $H(s)=1+0.12s$. In this case, the computed ITAE value is equal to 19.4×10^{-3}. Now, when the PMD-action feedback compensator of the form given by (3) is used for the control system, and the parameters h and k are adjusted until the minimum ITAE is obtained, it is found that the optimum values of h and k are 0.082 and 0.104, respectively, that is,

$$H(s)=1+1.2683(1-e^{-0.082s}), \tag{5}$$

for which the ITAE is 10×10^{-3}. For comparison, the step responses of the system equipped with optimally adjusted PD-action compensator and that of the optimally adjusted PMD-action compensator are plotted in Fig. 1(b). These plots and the ITAE indexes shown above suggest that the PMD-action compensator is definitely better than the conventional PD scheme. In addition, the system using PMD-action should be less sensitive to high-frequency disturbances as reported in [6]. It is remarked that the optimum values of h and k in (5) are found by an iterative method and with the aid of the map of contours of ITAE as shown in Fig. 2.

B. Determining Optimum Parameters for PMD-Action Compensators: A Rule of Thumb

In this section, it is reported that, to a certain extent, a design rule can be given for determining optimum parameters of the PMD-action compensators for a class of second-order dynamics systems.

As is well known [8], the dynamic system with a time-delay element in the feedback path is an infinite-dimensional system, containing infinite poles. In general, any analytical treatment of such a system seems rather difficult, especially when the system needs to be optimally designed. Our approach is to examine data on the optimum parameters of PMD-action compensators for various second-order plants, and find a simple rela-

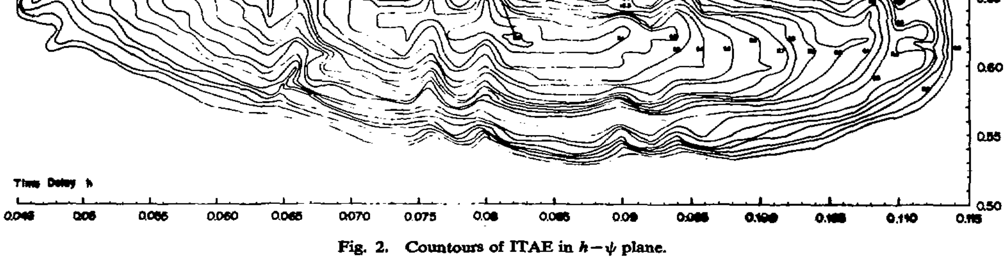

Fig. 2. Countours of ITAE in $h-\psi$ plane.

TABLE I

a/ω_0	ψ	λ	a/ω_0	ψ	λ
0	0.6	0.68	0.8	0.68	0.745
0.1	0.61	0.69	0.9	0.69	0.75
0.2	0.62	0.70	1.0	0.70	0.755
0.3	0.63	0.705	1.1	0.72	0.77
0.4	0.64	0.715	1.2	0.74	0.785
0.5	0.65	0.725	1.3	0.75	0.79
0.6	0.66	0.738	1.4	0.77	0.805
0.7	0.67	0.740			

tionship, if any, between the parameters of the plant and the compensator.

We assume that the plant to be controlled is described by (1) while the feedback compensator to be designed has the transfer function of (3). In this case, the overall transfer function of the control system is

$$\frac{C(s)}{R(s)}=\frac{\omega_0^2}{s^2+as+\frac{k\omega_0}{h}(1-\exp(-hs))+\omega_0^2}. \tag{5}$$

Examining data obtained by extensive digital computer simulations, we have found that there exists a strong and simple correlation between the plant parameter a/ω_0 and the compensator parameters k and h. To be specific, let us first use the relation $\exp(-sh)=\lim_{N\to\infty}(1-hs/N)^N$ to approximate the delay operator $\exp(-hs)$ [9] as

$$\exp(-hs)\doteq 1-hs+\frac{h^2s^2}{4}. \tag{6}$$

Then, the transfer function of (5) may be approximated as

$$\frac{C(s)}{R(s)}\approx\frac{\dfrac{\omega_0^2}{1-\dfrac{h\cdot k\cdot\omega_0^2}{4}}}{s^2+\dfrac{(a+k\omega_0^2)}{1-\dfrac{h\cdot k\cdot\omega_0^2}{4}}s+\dfrac{\omega_0^2}{1-\dfrac{h\cdot k\cdot\omega_0^2}{4}}} \tag{8}$$

or

$$\frac{C(s)}{R(s)}\approx\frac{\tilde{\omega}_0^2}{s^2+2\lambda\tilde{\omega}_0 s+\tilde{\omega}_0^2}$$

where the parameters λ and $\tilde{\omega}_0$ are defined by the relations

$$2\lambda\tilde{\omega}_0=\frac{a+k\omega_0^2}{1-\dfrac{k\omega_0^2h}{4}} \tag{9a}$$

$$\tilde{\omega}_0^2=\frac{\omega_0^2}{1-\dfrac{k\omega_0^2h}{4}}. \tag{9b}$$

For various values of the system parameter a, the optimum values of the compensators h and k were found by the same iterative method as described in Section II-A. These data are arranged in Table I in terms of λ and ψ where the parameter ψ is defined by the relation

$$2\psi\omega_0=a+k\omega_0^2. \tag{9c}$$

We found that there exist approximately linear relations between a/ω_0, ψ, and λ. To a certain extent, these relations between ψ, λ, and a/ω_0 can be modeled as

$$\psi=0.6+0.1(a/\omega_0) \tag{10a}$$

$$\lambda=0.085(a/\omega_0)+0.68. \tag{10b}$$

From (9) and (10), the parameters h and k of the feedback PMD-action compensator are determined as

$$k=\frac{1}{\omega_0}(1.2-0.8(a/\omega_0)) \tag{11a}$$

$$h=\frac{1}{\omega_0}\left[\frac{4}{1.2-0.8(a/\omega_0)}\right]\left[1-\frac{\{(0.6+0.1(a/\omega_0)\}^2}{\{(0.085(a/\omega_0)+0.68\}^2}\right]. \tag{11b}$$

Equation (11) is what we like to suggest as a practical design rule in determining the parameters of a PMD-action compensator (3) for the given plant dynamics (1) with $0<a/\omega_0<1.3$. To show the validity, ITAE indexes for various compensation schemes are plotted in Fig. 3 against the plant parameters. As is obvious from this graph, the compensator which is designed by the rule of (11) would be better than the corresponding PD-action type in the sense of ITAE.

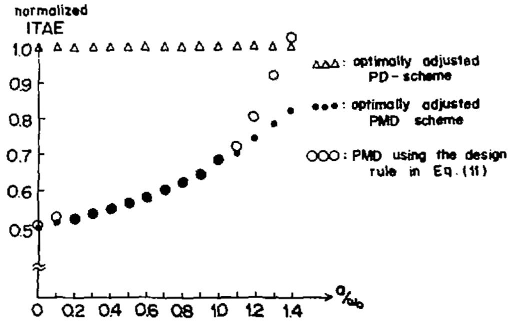

Fig. 3. Comparison of ITAE for various compensator schemes.

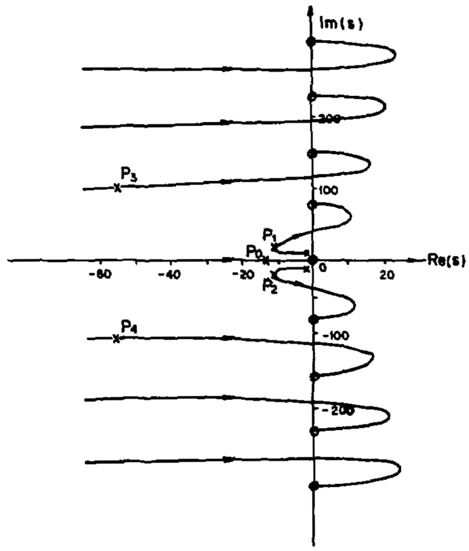

Fig. 4. Root-locus plot of the system in (4) with the optimal PMD-compensator.

It is particularly interesting to examine the root-loci of the closed-loop system. In Fig. 4, the root-loci of the system (4) for variable k and the closed-loop poles for the optimum parameter $k^*=0.104$ are sketched. The parameter h is fixed as $h^*=0.082$. When the three dominant poles p_0, p_1, and p_2 in Fig. 4 are used to approximate the system (4) with the optimal PMD-action compensator by a third-order ordinary linear model, the performances are found to be very close to each other. Note that the closed-loop pole configuration of p_0, p_1, and p_2 cannot be obtained by any PD-action feedback compensation, which partly explains the performance feature revealed in Fig. 1(b). Also observe that the closed-loop system with the PMD-action compensator is conditionally stable with the stable range $0<k\leq 0.633$. The root-loci in Fig. 4 is obtained by employing a technique similar to Pan and Chao's method [10].

If the dynamics of the controlled process are given in a general form

$$\overline{G_p(s)} = \frac{\omega_0^2}{s^2 + as + \omega_0^2} \tag{1$'$}$$

then a PMD-action compensator with the transfer-function of the form

$$\overline{H(s)} = \frac{k}{h}(1 - e^{-hs}) \tag{3$'$}$$

can be used in place of (3).

III. Concluding Remarks

It was shown that the PMD-action compensator for a second-order system was better than the conventional PD-type when the performance were compared in terms of the ITAE index values and a simple design rule was provided as (11). This rule (11) can be further improved by modeling the obtained data in Table I via piecewise linear lines. In fact, instead of (11), the following relationship may be used as a design a rule to obtain an improvement in the range of $1 < a/\omega_0 < 1.4$.

$$h^* = \frac{0.01}{\omega_0}\left(80\left(\frac{a}{\omega_0}\right)^2 + 74\right), \qquad 0 < \frac{a}{\omega_0} < 1.4$$

$$k^* = \begin{cases} \frac{1}{\omega_0}\left(1.2 - 0.8\left(\frac{a}{\omega_0}\right)\right), & 0 < \frac{a}{\omega_0} < 1 \\ \frac{1}{\omega_0}\left(1.05 - 0.65\left(\frac{a}{\omega_0}\right)\right), & 1 < \frac{a}{\omega_0} < 1.4. \end{cases}$$

For dynamic systems of third or higher order, the design of a compensator using PMD-action can be done but it seems that a general rule for an optimum compensator may be obtained only after extensive search.

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A Relation Between Overshoot and Sampling Period in Sampled Data Feedback Control Systems

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***Abstract*—This correspondence shows that the overshoot of response of a sampled data feedback control system sometimes tends to infinity as the sampling period tends to zero provided that the assigned poles are constant. This suggests that an optimal sampling period exists in designing deadbeat controllers.**

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I. Introduction

In the design of sampling data feedback control systems, sampling period has great influence on control effects. If the sampling period can be set to several different values or it can be set arbitrarily then it can be adjusted so that it minimizes some performance index of the control system. Sawaragi *et al.* [1] discussed such a problem using first- and second-orders servo systems and they showed that a nonzero optimal sampling period exists which minimizes integrated square error. This correspondence partially extends the results to multivariable systems and shows that the overshoot of the response sometimes tends to infinity as the sampling period tends to zero provided that the assigned poles are constant. This suggests that an optimal sampling period, which is not zero, exists in designing deadbeat controllers.

II. Statements of Problems

For a controllable single input time-invariant system

$$\dot{x}(t) = Ax(t) + bu(t), \qquad y(t) = cx(t):$$
$$A(n\times n),\ b(n\times 1),\ c(1\times n), \qquad t \geq 0,\ x(0) = x_0 \tag{1}$$

assume that the control input $u(t)$ is given by a piecewise constant state feedback

$$u(t) = -f(p)x(t_{i-1}): \qquad t_{i-1} < t < t_i \tag{2}$$

where $t_i = t_{i-1} + p$ and p is a constant sampling period.

If $x(t_i)$ and $y(t_i)$ are represented by x_i and y_i, respectively, the closed loop system is expressed by a sampled data feedback system

$$x_i = [F(p) - g(p)f(p)]x_{i-1}, \qquad y_i = cx_i:$$
$$x_0 = x(0), \qquad i = 1,2,3,\cdots \tag{3}$$

where $F(p)$ and $g(p)$ are given by

$$F(p) = \exp(Ap), \qquad g(p) = [\exp(Ap) - I]A^{-1}b. \tag{4}$$

In the case where $|A| = 0$, $g(p)$ can be determined from

$$g(p) = p(I + Ap/2! + A^2p^2/3! + \cdots)b. \tag{5}$$

It is well known [2] that the pair $[F(p), g(p)]$ is controllable for almost all sampling periods p provided that (A,b) is controllable. Therefore, the poles $\{r_i(p),\ i = 1 \sim n\}$ of the closed loop system, i.e., the eigenvalues of $F(p) - g(p)f(p)$, can be assigned freely.

The square error sum of the output y_i

$$J = \sum_{i=0}^{\infty} y_i^2 \tag{6}$$

can be represented by

$$J = x_0^T L x_0 \tag{7}$$

where L satisfies

$$L = \sum_{k=0}^{\infty} \{[F(p) - g(p)f(p)]^k\}^T c^T c [F(p) - g(p)f(p)]^k \tag{8}$$

provided that the system (3) is asymptotically stable [2]. Further, L is a positive definite matrix if $[c, F(p) - g(p)f(p)]$ is observable.

In the sequel, we discuss the behavior of J when the sampling period p tends to zero under a condition

$$r_i(p) = r_i = \text{constant} \quad \text{and} \quad |r_i| < 1, \qquad (i = 1 \sim n). \tag{9}$$

Such a problem, for example, occurs in discussing control effects of deadbeat controllers and in determining an optimal sampling period since, for this controller, r_i $(i = 1 \sim n)$ are fixed to zero. Usually we guess that $J \to 0$ as $p \to 0$, however, $J \to \infty$ as $p \to 0$ is true except in some special cases, and this suggests that an optimal sampling period exists which is