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# A Root-Locus Technique for Linear Systems with Delay

IL HONG SUH AND ZEUNGNAM BIEN

***Abstract*—A new method of plotting the root-loci is developed for the linear control system with delay in control or in state. In case of the system with delay in control, the root-locus plot starts from neighborhoods of the open-loop zeros instead of the open-loop poles and thus the effect of time-delay is easily handled. In case of the system with delay in state, the open-loop poles are firstly found by applying the root-locus method for the system with delay in control and then the desired root-loci are found by starting the root-loci plot from the open-loop poles.**

## I. INTRODUCTION

The root-locus method has been used as an invaluable design tool for linear feedback systems. Although the procedure of constructing the root-loci for finite dimensional linear system is well established, the root-locus plot for systems with time delay is not easily obtained because the solution of a trancendental equation is involved. For linear systems with time-delay in state variable, for example, no definite method is known to the authors. In case of feedback systems with time delay in control variable, several methods exist for computing the root locus plot such as branch following methods in [1], [2], and [3] or grid search method in [4]. One of the common features of these methods is that the root-loci start from the open-loop poles so that those branches which are not directly related to the open-loop poles may not be effectively constructed.

In this note, a new method of plotting the root-locus is suggested for the closed-loop systems with time-delay in control or in state. Based on the idea of Pan and Chao [5], the method when applied for the system with time-delay in control renders the branches starting from the neighborhood points of the open-loop zeros. Compared with the existing methods, this method reflects more clearly the influence of the time-delay term and reveals more about those branches not directly related to open-loop poles.

The root-locus plot of the system with delay in state variable is also obtained essentially in the same manner as in the case with delay in control.

In the sequal, $A$, $b$, and $c$ stand for $n \times n$, $n \times 1$, and $1 \times n$ constant matrices, respectively, and $x$, $y$, $u$, and $h$ denote $n$-dimensional state vector, scaler output, scaler input, and time delay, respectively.

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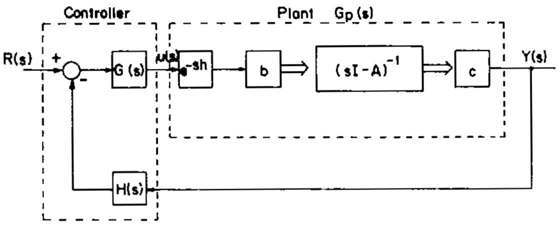


Fig. 1. Closed-loop system with delay in control.

## II. ROOT-LOCUS FOR SYSTEMS WITH TIME-DELAYS

### *A. Systems with Delay in Control*

Consider the linear system with delay in control shown in Fig. 1, whose dynamics is given by

$$\dot{x} = Ax + bu(t-h)$$
$$y = cx. \tag{1}$$

In this case, the plant transfer function $G_p(s)$ is

$$G_p(s) = c(Is - A)^{-1} b e^{-sh}. \tag{2}$$

Let the open-loop transfer function of the control system be given by

$$G_c(s)G_p(s)H(s) = K\frac{N(s)}{D(s)}e^{-sh} \tag{3}$$

where $K$ is the open-loop gain, $D(s)$ and $N(s)$ are polynomial functions of $s$ of degree $n$ and $m$, respectively. It is desired to plot the locus of the poles of the closed-loop transfer function

$$T(s) = \frac{G_c(s)G_p(s)}{1 + G_c(s)G_p(s)H(s)} = \frac{D(s)G_c(s)G_p(s)}{D(s) + KN(s)e^{-sh}} \tag{4}$$

as $K$ varies from zero to infinity.

Note that, for each $K$, the characteristic equation of the control system

$$g(s,K) \triangleq D(s) + KN(s)e^{-sh} = 0 \qquad 0 \leq K < \infty \tag{5}$$

is a transcendental equation in $s$ and thus may include an infinite number of roots. Therefore, the number of root-locus branches of (5) as $K$ varies from 0 to $\infty$ is infinite. If such an infinite number of branches must be determined for the design of control systems involving time delay, the root locus method would be an impractical tool. Fortunately, it is known that the number of zeros of $g(s, K)$ each of whose real part is greater than any given real number is finite if $N(s)/D(s)$ is strictly proper rational [6], and that all the zeros of $g(s, K)$ except some finite number around the origin lie in the left half of $s$-plane [8]. Thus, most of the zeros of $g(s, K)$, being located far from the imaginary axis, do not contribute much in the system performance, and so only a finite number of root-loci near the origin may be needed in determining the characteristics of the closed-loop system with delay as depicted in Fig. 1.

The technique of constructing root-loci developed by Pan and Chao [5], which was proposed to handle the finite dimensional systems is extended in the following to solve (5).

First, observe that solving (5) is equivalent to solving the equation

$$f(s, \hat{K}) = \hat{K}D(s) + N(s)e^{-sh} = 0, \qquad 0 \leq \hat{K} < \infty. \tag{6}$$

As in [5], introduce a new independent variable "$t$" and show that the problem of finding the roots of $f(s, \hat{K}) = 0$ in (6) for each $\hat{K}$ is equivalent to the problem of solving the following simultaneous nonlinear differential equations:

$$\frac{ds}{dt} = -\frac{N(s)e^{-sh} + \hat{K}D(s) \pm D(s)}{(N'(s) - hN(s))e^{-sh} + \hat{K}D'(s)}, \qquad s(0) = s_0 \tag{7-1}$$

$$\frac{d\hat{K}}{dt} = 1, \qquad \hat{K}(0) = \hat{K}_0. \tag{7-2}$$

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Here $N'(s) \triangleq dN(s)/ds$ and $D'(s) \triangleq dD(s)/ds$. Also $s_o$ is a root of (6) for an initial gain $\hat{K}_o$. Once the initial values $\hat{K}_o$ and $s_o$ are known, then the trajectories $s(t)$ and $\hat{K}(t)$, which are the solutions of (6), are obtained by a numerical integration. Thus, it remains to determine the initial value of $s_o$ for a given $\hat{K} = K_o$ satisfying (6), i.e., it needs to solve

$$f(s, \hat{K})|_{\hat{K}=\hat{K}_o} = 0. \tag{8}$$

For this, observe that

$$f(s, \hat{K}) = \hat{K}D(s) + \gamma N(s) + N(s)(e^{-sh} - \gamma) \tag{9}$$

where $\gamma$ is a real constant. If we define $T(s)$ as

$$T(s) \triangleq \frac{\hat{K}}{1 + \hat{K}\dfrac{D(s) + \gamma N(s)/\hat{K}_o}{N(s)(e^{-sh)} - \gamma)}}, \tag{10}$$

it is obvious that the poles of $T(s)$ in (10) when $\hat{K} = \hat{K}_o$ are the same as the roots of (8) and can be obtained by plotting the locus of the poles of $T(s)$ as $\hat{K}$ varies from zero to $\hat{K}_o$. The root-locus of (10) can be obtained as in [5] by solving the following nonlinear differential equations:

$$\frac{ds}{dt} = -\frac{N(s)(e^{-sh} - \gamma) + \hat{K}(D(s) + \gamma N(s)/\hat{K}_o) \mp (D(s) + \gamma N(s)/\hat{K}_o)}{(N'(s) - hN(s))e^{-sh} + \hat{K}D'(s) + \gamma(\hat{K}/\hat{K}_o - 1)N'(s)},$$

$$s(0) \triangleq \text{roots of } N(s) = 0, \quad \text{or } e^{-sh} - \gamma = 0, \tag{11}$$

and

$$\frac{d\hat{K}}{dt} = \pm 1, \qquad \hat{K}(0) = 0. \tag{12}$$

It is observed that, while the initial conditions $s(0)$ and $\hat{K}(0)$ in (7) were functionally related by (8), the initial conditions for (11) and (12) are independently given so that $s(t)$ and $K(t)$ readily obtained by numerical integration. Here $\gamma$ is usually chosen to be 1 or $-1$ since the roots of $e^{-sh} \mp 1 = 0$ are easily found.

As commented in [5], when the denominator of the right-hand side of (7) or (11) becomes zero at a point $s^*$ for some $\hat{K}$, it is noted that (7) or (11) is not valid. Such a point $s^*$ is called a singular point [5], and may exist when

$$\left.\frac{d\hat{K}}{ds}\right|_{s=s^*} = 0, \quad \text{or} \quad \left.\frac{\partial f(s, \hat{K})}{\partial s}\right|_{s=s^*} = 0. \tag{13}$$

To handle this singular case, the results on the characterization of singular points in [5] are extended as follows.

*Theorem 1:* Let

$$f(s, \hat{K}) = \hat{K}D(s) + N(s)e^{-sh}$$

where the degrees of polynomials $D(s)$ and $N(s)$ are $n$ and $m$, respectively, with $n \geq m$. Then the number of roots with multiplicity $N$ of the equation $f(s, \hat{K}) = 0$ for fixed $\hat{K}$ are at most $n + m + 2 - N$, where

$$2 \leq N \leq n + m + 1. \tag{14}$$

The proof of Theorem 1 is given in the Appendix. It is easily verified from Theorem 1, that for each $\hat{K}$, the transcendental equation $f(s, \hat{K}) = 0$ has at most a finite set of multiple roots, and if they exist, the multiplicity is also finite. Based on the above theorem and the results on the singular points characterized by the properties of higher order derivatives given in [5], the following corollary is deduced.

*Corollary:* Suppose, for some $\hat{K}$, $s^*$ is a singular point of the root-locus of (6) such that $f(s^*, \hat{K}) = 0$ and

$$\left.\frac{\partial^l f(s, \hat{K})}{\partial s^l}\right|_{s=s^*} = 0, \qquad l = 1, 2, \cdots, N-1, \tag{15}$$

but

$$\left.\frac{\partial^N f(s, \hat{K})}{\partial s^N}\right|_{s=s^*} \neq 0, \qquad 2 \leq N \leq n + m - 1. \tag{16}$$

Then the root-locus plot of $f(s, \hat{K}) = 0$ contains $N$ branches intersecting at $s = s^*$.

The proof is similar to the one in [5, p. 858] and hence omitted. It follows from the above corollary that the solutions of (7) can be determined by (15) and (16) whether they are singular points or not. Therefore, if the computed zero is found to be a singular point, some modifications must be made as in [5] when plotting the root-locus as $\hat{K}$ increases as follows:

$$s_{\text{new}} = \begin{cases} s_{\text{old}} + \Delta s & \text{for odd } N \\ s_{\text{old}} + \Delta s e^{-j\pi/N} & \text{for even } N. \end{cases} \tag{17}$$

Here $\Delta s$ is a sufficiently small vector which is tangential to the locus at the singular point, and $s_{\text{old}}$ denotes the singular point.

*B. Systems with Delay in State*

In this subsection, it is shown that the method developed in Section II-A can be used to obtain the root-locus plot of a class of system with delay in state.

Consider a single-input, single-output linear control system with delay in state whose dynamics is given by

$$\dot{x} = A_o x + A_1 x(t-h) + bu$$
$$y = cx. \tag{18}$$

It is assumed that the plant transfer function $G_p(s) \triangleq y(s)/u(s)$ can be expressed in the form of

$$G_p(s) = \frac{R_1(s) + R_2(s)e^{-sh}}{P_1(s) + P_2(s)e^{-sh}} \tag{19}$$

where $P_1(s)$, $P_2(s)$, $R_1(s)$, and $R_2(s)$ are polynomials of $s$. Such a case occurs, for example, when $A_1$ has only one nonzero element. First consider the case when $P_2(s)$ is not identically equal to zero so that the open-loop transfer-function of the control system is given as

$$G_c(s)G_p(s)H(s) = K\frac{\pi(s)}{D(s) + N(s)e^{-sh}}. \tag{20}$$

Here $K$ is the open-loop gain, $D(s)$ and $N(s)$ are some polynomial functions of $s$, and $\pi(s)$ is of the form

$$\pi(s) = D_1(s) + N_1(s)e^{-sh}. \tag{21}$$

It is desired to plot the poles as $K$ varies for the closed-loop transfer-function

$$T(s) = \frac{G_c(s)G_p(s)}{1 + K\dfrac{\pi(s)}{D(s) + N(s)e^{-sh}}}. \tag{22}$$

The desired root-loci of the characteristic equation for system (22) can be obtained by solving the following set of nonlinear differential equations:

$$\frac{ds}{dt} = -\frac{D(s) + K\pi(s) + N(s)e^{-sh} \mp \pi(s)}{D'(S) + [N'(s) - hN(s)]e^{-sh} + K\pi'(s)}, \qquad s(0) = s_o \tag{23}$$

where

$$\pi'(s) \triangleq \frac{d\pi(s)}{ds},$$

and

$$\frac{dK}{dt} = \pm 1, \qquad K(0) = K_o. \tag{24}$$

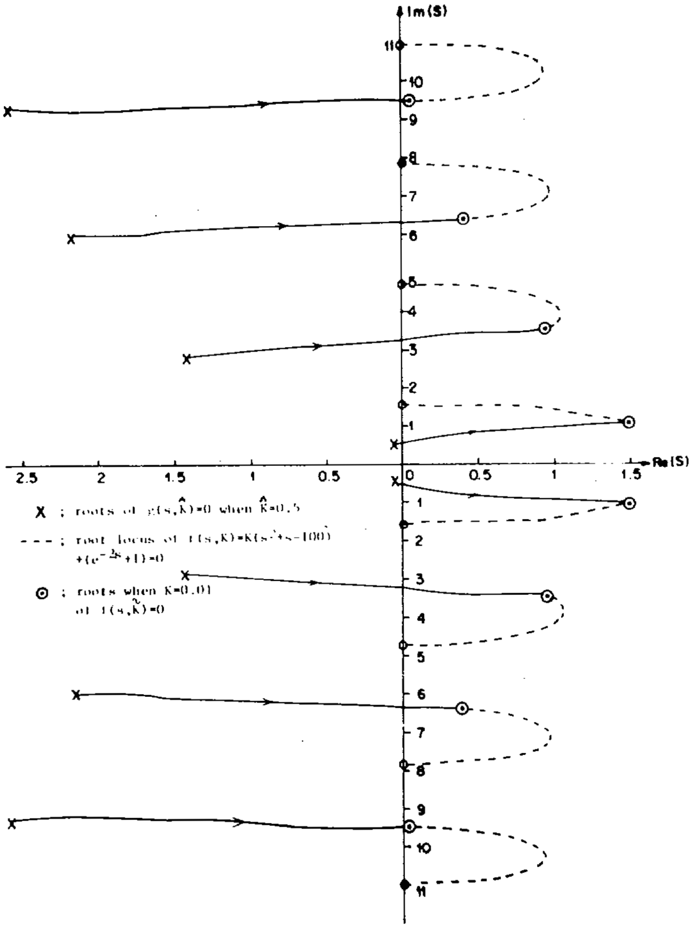


Fig. 2. Root-locus of $g(s,\hat{K})=s^2+s+\hat{K}e^{-2s}=0$, $0<\hat{K}\leqslant 100$.

For simplicity, the initial starting point for (24) is chosen to be zero, i.e., $K(0)=K_o=0$. Then the initial starting point $s(0)=s_o$ for (23) can be found by solving the equation

$$q(s)\triangleq D(s)+N(s)e^{-sh}=0. \tag{25}$$

Here the roots of (25) can be found by applying the root-locus method developed in Section II-A.

To handle the case when $P_2(s)$ in (19) is equal to zero, the closed-loop transfer-function in (22) is rearranged as follows: with $\hat{K}=1/K$,

$$\hat{T}(s)=\frac{G_c(s)G_p(s)/\pi(s)}{1+\hat{K}\dfrac{D(s)}{D_1(s)+N_1(s)e^{-sh}}}. \tag{26}$$

Then the root-locus of the characteristic equation of the system in (26) can be obtained as before by replacing $D(s)$, $N(s)$, $\pi(s)$, and $K$ in (23) with $D_1(s)$, $N_1(s)$, $D(s)$, and $\hat{K}$ in (26), respectively. It is remarked that, as in the case of the systems with delay in control, only some finite number of branches of root-locus needs to be considered for the design of the systems with delay in state if the degree of $D(s)$ is greater than that of $N(s)$ and $N_1(s)$, and if $\lim_{y\to\infty}|\pi(jy)|/|D(jy)+N(jy)e^{-jyh}|=0$.

## III. An Example

To show the use of the root-locus technique developed in the note, a simple example is now presented.

Consider the linear control system with delay in state whose open-loop transfer function is given by

$$G_c(s)G_p(s)H(s)=K\frac{0.5}{s^2+s+0.5e^{-2s}}. \tag{27}$$

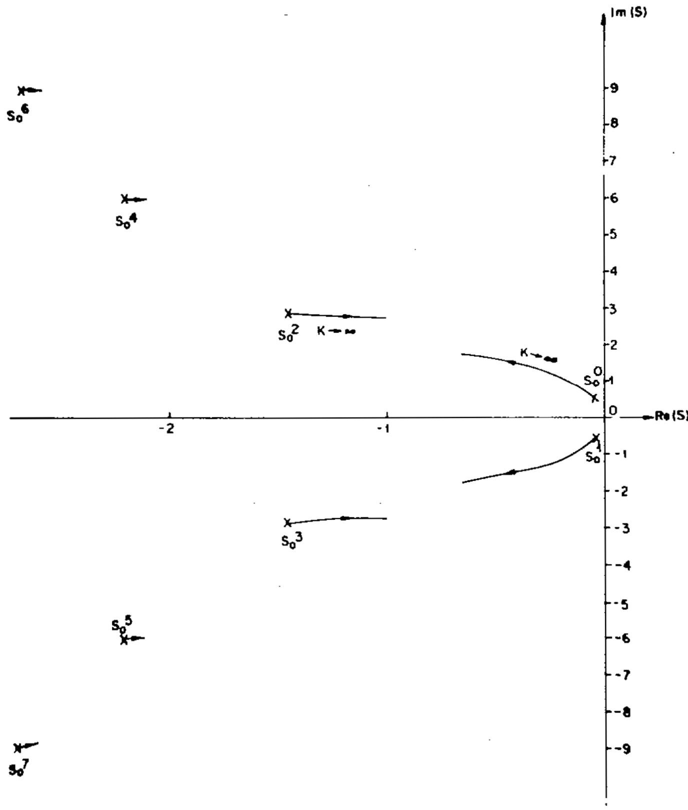


Fig. 3. Root-locus of $s^2-s+0.5e^{-2s}+0.5K=0$, $0\leqslant K<\infty$.

Then closed-loop transfer-function $T(s)$ is

$$T(s)=\frac{0.5}{s^2+s+0.5e^{-2s}+0.5K}. \tag{28}$$

Before solving (23), it is necessary to determine initial starting point. Let $K(0)=0$. The initial starting point $s_o$ at $K_o=0$ in (24) is obtained by solving the equation

$$q(s)=s^2+s+0.5e^{-2s}=0. \tag{29}$$

For this, (29) is now considered as the characteristic equation of the system with delay in control whose transfer-function is given by

$$\tilde{T}(s)=\frac{1}{1+\hat{K}\dfrac{e^{-2s}}{s(s+1)}} \tag{30}$$

with $\hat{K}=0.5$. But the root-locus of (30) for $0<\hat{K}\leqslant 100$ is easily obtained by the method is Section II-A, where $\gamma$ is chosen to be $-1$, and is sketched in Fig. 2. The roots at $\hat{K}=0.5$ are the initial starting points for the root-loci of (28) as $K$ varies from zero to infinity. The root-locus of (28) is obtained via (24) and sketched in Fig. 3 with the starting points denoted as $s_o^i$, $i=0,1,\cdots,7$.

## IV. Concluding Remarks

A root-locus technique was developed for the linear control systems with delay in control or in state by modifying Pan and Chao's method in [5]. The technique may be extended to the system with multiple delays in control and/or in state, and is found to be particularly useful in designing controllers with intentional time-delay [7].

## Appendix

*Proof of Theorem 1:* Let $\hat{K}D(s)\triangleq E(s)$, and denote $f(s,\hat{K})=\hat{f}(s)$ for fixed $\hat{K}$. If $s$ is a multiple root of $\hat{f}(s)$, whose order is greater or equal to 2, then $\hat{f}(s)=0$ and $\hat{f}'(s)=0$. Eliminating $e^{-sh}$ from these two equations, one finds that

$$[N'(s)-hN(s)]E(s)-E'(s)N(s)=0. \qquad \text{(A-1)}$$

Since (A-4) is a polynomial of $(n+m)$th-order, the number of multiple roots of order two or greater than two are at most $n+m$. Note that

$$[N(s)e^{-sh}]^{(l)}=\left[\left(\frac{d}{ds}-h\right)^{l}N(s)\right]e^{-sh}. \qquad \text{(A-2)}$$

Now consider the $(n-1)$th differentiation of $\hat{f}(s)$. Differentiating $\hat{f}(s)$ $(n-1)$ times gives rise to

$$\hat{f}^{(n-1)}(s)=f_1(s)+\left[\left(\frac{d}{ds}-h\right)^{n-1}N(s)\right]e^{-sh} \qquad \text{(A-3)}$$

where $f_1(s)$ is a first-order polynomial. If $s$ is a multiple root of multiplicity $(n+1)$ or greater than $(n+1)$, then $\hat{f}^{(n-1)}(s)=0$ and $\hat{f}^{(n)}(s)=0$. Eliminating $e^{-hs}$ from these two equations, one obtains

$$f_1^*(s)\left[\left(\frac{d}{ds}-h\right)^{n}N(s)\right]-\left(\frac{d}{ds}-h\right)^{n-1}N(s)=0 \qquad \text{(A-4)}$$

where $f_1^*(s)$ is a first-order polynomial. Since, for any integer $l$, $((d/ds)-h)^l N(s)$ is an $m$th-order polynomial, (A-4) is an $(m-1)$th-order polynomial. Thus, the number of multiple points of multiplicity $(n+1)$ or greater than $(n-1)$ are at most $(m+1)$. Consider $(n+1)$-times differentiation of $f(s)$, i.e.,

$$\hat{f}^{(n-1)}(s)=\left[\left(\frac{d}{ds}-h\right)^{n+1}N(s)\right]e^{-sh}. \qquad \text{(A-5)}$$

Suppose $s$ is a root of (A-5). Then $\hat{f}^{(n+1)}(s)=0$. If $s$ is a multiple roots of order $(n+3)$ greater than $(n+3)$, $\hat{f}^{(n+2)}(s)=0$, which gives

$$\hat{f}^{(n+2)}(s)=\left[\frac{d}{ds}\left\{\left(\frac{d}{ds}-h\right)^{n+1}N(s)\right\}\right]e^{-sh}-h\left(\frac{d}{ds}-h\right)^{n+1}N(s)e^{-sh}$$

$$=\left[\frac{d}{ds}\left\{\left(\frac{d}{ds}-h\right)^{n+1}N(s)\right\}\right]e^{-sh}=0. \qquad \text{(A-6)}$$

Clearly (A-6) is an $(m-1)$th-order polynomial. Thus multiple roots of multiplicity $(n+3)$ greater than $(n+3)$ are at most $(m-1)$. In a similar way, if $s$ is a multiple root of multiplicity $(n+m+2)$, then $\hat{f}^{(n+m+1)}(s)=0$, which gives

$$\frac{d^m}{ds^m}\left[\left(\frac{d}{ds}-h\right)^{n+1}N(s)\right]=\text{nonzero constant}. \qquad \text{(A-7)}$$

Thus, from (A-7), one may conclude that there exists no multiple root of multiplicity $(n+m+2)$ satisfying $\hat{f}(s)=0$. This completes the proof.

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# Uniform Controllability of a Class of Linear Time-Varying Systems

G. KERN

*Abstract*—**An applicable criteria for uniform complete controllability to a class of linear time-varying systems is presented.**

## I. Introduction

The problem of stabilizing "uniformly controllable" finite dimensional linear time-varying systems has been studied in various papers [1], [2]. We note that in the application of these results the problem of deciding whether a prescribed pair $(A(t), B(t))$ is uniformly completely controllable is often difficult, since it may require calculation of the transition matrix. In order to apply the results in stability analysis or system synthesis, it is useful to have criteria for uniform complete controllability which do not require calculation of the transition matrix. Silverman and Anderson [3] gave such criteria, which are also applicable in other problems which involve uniform complete controllability, but the constraints on the pair $(A(t), B(t))$ are quite restrictive. A weaker condition under which the results hold are presented only for single-input systems. In this paper we present a broad class of linear time-varying systems for which the criteria for uniform complete controllability are applicable, because the Gramian can be computed without knowledge of the transition matrix of the time-varying part.

## II. System Description and Results

Consider the linear time-varying system

$$\dot{x}=A(t)x-B(t)u \qquad (1)$$

where $x(t)$, an $n$-vector, is the state of the system at time $t$, and $u(t)$, an $r$-vector, is the input. The matrices $A(t)$ and $B(t)$ are of appropriate dimensions and their elements are piecewise continuous functions.

It will be assumed that system (1) is a bounded realization, that is, there exists a constant $K$ such that for all $t$

$$|A(t)|\leq K, \quad |B(t)|\leq K.$$

For any fixed $s\in J$, where $J$ is the interval $t_0\leq t\leq t_1$, we can write (1) in the form

$$\dot{x}=A(s)x+[A(t)-A(s)]x+B(t)u. \qquad (1')$$

Suppose that the reduced system

$$\dot{x}=A(s)x+B(t)u \qquad s\in J \text{ fixed}. \qquad (2)$$

is controllable over some interval $[t_0, t_1]$; thus, the system (2) can be driven from any initial state $x_0$ at time $t_0$ to the final state $x_1$ at time $t_1$; or, equivalently [4], suppose that the symmetric matrix

$$W(t_0,t_1)=\int_{t_0}^{t_1}e^{A(s)(t_1-\tau)}B(\tau)B^T(\tau)e^{A^T(s)(t_1-\tau)}d\tau \qquad (3)$$

is nonsingular. Then we can deduce that if the matrix $A(t)$ satisfies a Lipschitz condition, the original system (1′) is also controllable.

*Theorem 1:* Suppose $W(t_0, t_1)$ is nonsingular, and suppose the matrix $A(t)$ satisfies a Lipschitz condition

(i) $|A(t)-A(t')|\leq L|t-t'|$ for all $t, t'\in J$.

Then the system (1′) is completely controllable at time $t_0$.

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