are introduced. The problem considered below is the robustness of the configuration of Fig. 1(b) when the plant is subjected to linear feedback perturbations of the type discussed above and the linear feedback F is simultaneously replaced by $F+N$ where N is a memoryless map of Y into itself of the form

$$N = N_1 + N_2 \tag{10}$$

where N_1 and N_2 map Y_0 into itself, N_1 has finite incremental gain k_1 such that

$$\|N_1 y - N_1 z\| \leq k_1 \|y - z\| \qquad \forall y, z \in Y_0 \tag{11}$$

and $N_1 0 = 0$ and N_2 is bounded in the sense that there exists a scalar $q \geq 0$ such that $N_2 Y \subset Y_0$ and

$$\|N_2 y\| \leq \frac{q}{2} \qquad \forall y \in Y. \tag{12}$$

Theorem 2: If the feedback system of Fig. 1(b) is stable, then the system is also stable in the presence of simultaneous linear feedback perturbations of the plant and nonlinear perturbations of the feedback dynamics if conditions i) and ii) of Theorem 1 are satisfied and also

$$\mu \triangleq (1-\lambda)^{-1} \|L_c\| k_1 < 1. \tag{13}$$

Moreover, under these conditions, the responses y and y' of the real and perturbed feedback systems are related by the inequality

$$\|y' - y\| \leq \left\{ \frac{\mu}{1-\mu} + \lambda \right\} \frac{1}{(1-\lambda)} \|y\| + \|L_c\| \frac{q}{2} \frac{1}{(1-\lambda)(1-\mu)}. \tag{14}$$

Proof: The feedback relations of the perturbed system are

$$\begin{aligned} y' &= G(u - v), \qquad v = Hy', \\ u &= Ke \qquad e = r - Fy' - Ny' \end{aligned} \tag{15}$$

which can be written as

$$y' = L_c(r - K^{-1}Hy' - Ny') \tag{16}$$

or, bearing in mind the fact that (4) implies that $(I + L_c K^{-1} H)^{-1}$ exists in Y_0,

$$y' = \left(I + L_c K^{-1} H\right)^{-1} L_c(r - Ny'). \tag{17}$$

This equation has a unique solution $y' \in Y_0$ for each choice of $r \in Y_0$ if $(I + L_c K^{-1}H)^{-1} L_c N$ is a contraction. Taking initially the case of $N_2 = 0$ (ie., $q = 0$), this is clearly the case with contraction constant μ as

$$\begin{aligned} &\left\|\left(I + L_c K^{-1} H\right)^{-1} L_c(Ny - Nz)\right\| \\ &\qquad \leq \left\|\left(I + L_c K^{-1} H\right)^{-1}\right\| \cdot \|L_c\| k_1 \|y - z\| \\ &\qquad \leq \frac{\|L_c\| k_1}{1-\lambda} \|y - z\| \end{aligned} \tag{18}$$

using (4). We can in fact verify (14) in this case by choosing the first iterate $y_0' = 0$ in the successive approximation sequence and note that $y_1' = (I + L_c K^{-1} H)^{-1} L_c r = (I + L_c K^{-1} H)^{-1} y$ to give

$$\left\|y' - \left(I + L_c K^{-1} H\right)^{-1} y\right\| \leq \frac{\mu}{1-\mu} \left\|\left(I + L_c K^{-1} H\right)^{-1} y\right\|. \tag{19}$$

Using the relationship $\|(I + L_c K^{-1} H)^{-1}\| \leq (1-\lambda)^{-1}$ and the triangle inequality then yields

$$\begin{aligned} \|y' - y\| &\leq \left\|y' - \left(I + L_c K^{-1} H\right)^{-1} y\right\| + \left\|\left(I + L_c K^{-1} H\right)^{-1} L_c K^{-1} H y\right\| \\ &\leq \frac{\mu}{(1-\mu)} \frac{1}{(1-\lambda)} \|y\| + \frac{\lambda}{(1-\lambda)} \|y\| \\ &= \left(\frac{\mu}{1-\mu} + \lambda\right) \frac{1}{(1-\lambda)} \|y\|. \end{aligned} \tag{20}$$

The case of $N_2 \neq 0$ is treated by noting that it is equivalent to the situation when $r \in Y_0$ is replaced by $r - N_2 y' \in Y_0$. Stability is therefore unaffected by N_2 and (20) holds with y replaced by $y - L_c N_2 y'$, i.e., we obtain

$$\begin{aligned} \|y' - y\| &\leq \|y' - (y - L_c N_2 y')\| + \|L_c N_2 y'\| \\ &\leq \left(\frac{\mu}{1-\mu} + \lambda\right) \frac{1}{1-\lambda} \|y - L_c N_2 y'\| + \|L_c\| \frac{q}{2} \\ &\leq \left(\frac{\mu}{1-\mu} + \lambda\right) \frac{1}{1-\lambda} \left\{ \|y\| + \|L_c\| \frac{q}{2} \right\} + \|L_c\| \frac{q}{2} \end{aligned} \tag{21}$$

which is simply (14). This completes the proof of the theorem.

Remark 5: If, as is usually the case in nonlinear studies, a causality structure exists, the result holds with all vectors and operators replaced by their truncated counterparts. The proof is trivial.

In conclusion, a partial answer to the problem of robustness of feedback systems to unstable plant perturbations has been provided by considering stable linear feedback perturbations to the plant. The results are a generalization of those underlying known design techniques [14] and also enable the effect of nonlinear feedback perturbations on both stability and response characteristics to be assessed.

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A Note on the Stability of Large Scale Systems with Delays

IL HONG SUH AND ZEUNGNAM BIEN

***Abstract*—A stability test is proposed for large scale systems with delays by employing both the aggregation technique based on a Lyapunov function and the strictly quasi-diagonal dominance.**

I. Introduction

Recently, there are a number of research works on the stability of the large scale systems. Callier *et al.* obtained a necessary and sufficient condition for the large scale linear distributed systems to be stable by

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using the concept of input output stability incorporated with the hierarchical decomposition technique [1]. Siljak [2] derived a sufficient condition for the large scale systems with memoryless nonlinear interactions to be stable by employing the aggregation method based on the Lyapunov function, where the aggregated system was assumed to be a linear autonomous system with no delay. In [3], Moylan and Hill tried to unify the approach of input–output stability and the approach based on the Lyapunov theory by developing the concept of dissipativeness. In [4] and [5], it was found that the time delays cause no particular difficulties in determining the stability of a specific type of large scale systems with delays in interactions.

In this note, a stability test is proposed for large scale linear systems with delays, which is applicable for a more general class of large scale systems than those considered in [4] and [5]. For this, the aggregated system considered in the note is chosen to be a linear autonomous system with delays.

Throughout the note, A^T and $\|a\|$ will denote the transpose of a matrix A and the Euclidian norm of a finite-dimensional vector a, respectively.

II. Stability of a Large Scale System with Delays

Consider a large scale system with delays in state and interconnections whose dynamics is given by

$$\dot{x}_i(t)=A_ix_i(t)+\hat{A}_ix_i(t-T_i)+\sum_{\substack{j=1\\ j\neq 1}}^{N}A_{ij}x_j(t-h_{ij}),\qquad i=1,2,\cdots,N \tag{1}$$

where $x_i\in R^{n_i}$ is a state vector, A_i is a stable $n_i\times n_i$ constant matrix, $\hat{A}_i$ is an $n_i\times n_i$ constant matrix, A_{ij} is an $n_i\times n_j$ constant interconnection matrix, and $h_{ij}\geqslant 0$ and $T_i\geqslant 0$ denote time delay. Let G_i be a positive definite symmetric matrix, and let H_i be a positive symmetric matrix such that

$$A_i^TH_i+H_iA_i=-G_i. \tag{2}$$

It is noted that since A_i is assumed to be stable, for any positive definite matrix G_i, there always exist a positive definite matrix H_i satisfying (2) [2]. As in [8], let v_i be a positive scaler function given by

$$v_i(t)=\left(\|H_i^{-1}\|x_i^T(t)H_ix_i(t)\right)^{1/2}\qquad i=1,2,\cdots,N. \tag{3}$$

From (1), consider the interaction-free system with $\hat{A}_i=0$ given by

$$\dot{x}_i(t)=A_ix_i(t),\qquad i=1,2,\cdots,N. \tag{4}$$

Then it is easily shown from (3) and (4) that, for any t

$$\|x_i(t)\|\leqslant v_i(t)\leqslant \eta_{i_1}\|x_i(t)\|, \tag{5}$$

$$\frac{\|H_i^{-1}\|}{2}v_i^{-1}(t)\left(x_i^T(t)G_ix_i(t)\right)\geqslant \eta_{i_2}v_i(t), \tag{6}$$

and

$$\|\text{grad } v_i(t)\|\leqslant \eta_{i_1} \tag{7}$$

where $\eta_{i_1}=\sigma_M^{1/2}(H_i)/\sigma_m^{1/2}(H_i)$ and $\eta_{i_2}=\sigma_m(G_i)/2\sigma_M(H_i)$. Here $\sigma_m(H_i)$ and $\sigma_M(H_i)$ denote the minimum and maximum eigenvalue of H_i, respectively. Let ξ_{ij} and π_i be defined by

$$\xi_{ij}=\sigma_M^{1/2}\left(A_{ij}^TA_{ij}\right) \tag{8}$$

and

$$\pi_i=\sigma_M^{1/2}\left(\hat{A}_i^T\hat{A}_i\right). \tag{9}$$

It is now shown that the time-delay actions cause no difficulties in determining the stability of the system in (1).

Theorem 1: Consider a large scale system described by (1). Let A_i be a stable matrix for $i=1,2,\cdots,N$. Let $N\times N$ constant matrix $W=(\omega_{ij})$ be given by

$$\omega_{ij}=-(\eta_{i_2}-\eta_{i_1}\pi_i)\delta_{ij}+\eta_{i_1}\xi_{ij}(1-\delta_{ij}) \tag{10}$$

where δ_{ij} denotes Kronecker delta symbol, and η_{i_1}, η_{i_2}, π_i, and ξ_{ij} are given in (5)–(9), respectively. If $\omega_{ii}<0$ and W is strictly quasi-diagonal dominant, then (1) is stable.

For the Proof of the Theorem 1, we first consider the stability of a specific class of linear autonomous system with delays whose dynamics is given by

$$\dot{z}_i(t)=-a_{ii}z_i(t)+b_{ii}z_i(t-T_i)+\sum_{\substack{j=1\\ j\neq i}}^{N}a_{ij}z_j(t-h_{ij}),\qquad i=1,2,\cdots,N \tag{11}$$

where $a_{ii}>0$, b_{ii} and a_{ij} are real constants, and T_i and h_{ij} are positive real constants which denote the time delay. We propose a sufficient condition for the stability of (11) in the following:

Lemma 1: If there exist positive scalers d_i, $i=1,2,\cdots,N$, such that for all i

$$d_ia_{ii}>d_i|b_{ii}|+\sum_{\substack{j=1\\ j\neq i}}^{N}d_j|a_{ji}|, \tag{12}$$

then (10) is stable.

The proof of Lemma 1 is straightforward if the Nyquist array technique in [7] is employed. Hence, the proof is omitted. It is remarked that in [4],[5], a sufficient condition for the stability of the case when $b_{ii}=0$ in (11) was given as in Lemma 1.

Proof of Theorem 1: To determine the stability of (1), let a positive scaler function v_i in (2) be Lyapunov functions for (1). Then it is easily shown that for $i=1,2,\cdots,N$ and for all $t\in R$,

$$\begin{aligned}\dot{v}_i(t)=&-\frac{1}{2}v_i^{-1}(t)\|H_i^{-1}\|\left(x_i^T(t)G_ix_i(t)\right)\\&+\left(v_i^{-1}(t)\|H_i^{-1}\|H_ix_i(t)\right)^T\hat{A}_ix(t-T_i)\\&+\left(v_i^{-1}(t)\|H_i^{-1}\|H_ix_i(t)\right)^T\sum_{\substack{j=1\\ j\neq i}}^{N}A_{ij}x_j(t-h_{ij})\end{aligned} \tag{13}$$

$$\leqslant -\eta_{i_2}v_i(t)+\eta_{i_1}\|\hat{A}_ix_i(t-T_i)\|+\eta_{i_1}\sum_{\substack{j=1\\ j\neq i}}^{N}\|A_{ij}x_j(t-h_{ij})\| \tag{14}$$

$$\leqslant -\eta_{i_2}v_i(t)+\eta_{i_1}\pi_i\|x_i(t-T_i)\|+\eta_{i_1}\sum_{\substack{j=1\\ j\neq i}}^{N}\xi_{ij}\|x_j(t-h_{ij})\| \tag{15}$$

$$\leqslant -\eta_{i_2}v_i(t)+\eta_{i_1}\pi_iv_i(t-T_i)+\eta_{i_1}\sum_{\substack{j=1\\ j\neq i}}^{N}\xi_{ij}v_j(t-h_{ij}). \tag{16}$$

From (16), let the aggregated system of (1) be given by

$$\dot{q}_i(t)=-\eta_{i_2}q_i(t)+\eta_{i_1}\pi_iq_i(t-T_i)+\eta_{i_1}\sum_{\substack{j=1\\ j\neq i}}^{N}\xi_{ij}q_j(t-h_{ij}),\qquad i=1,2,\cdots,N. \tag{17}$$

It is shown in [2] that by the comparison principle, if (17) is stable, then (1) is stable. In (17), by letting $a_{ii}=\eta_{i_2}$, $b_{ii}=\eta_{i_1}\pi_i$, and $a_{ij}=\eta_{i_1}\xi_{ij}$, and by Lemma 1, we can conclude that if W is strictly quasi-diagonal dominant, (1) is stable. This completes the proof.

It is remarked that stability condition given in Theorem 1 is the same as the stability condition for the case when $h_{ij}=0$ and $\hat{A}_i=0$ for $i,j=1,2,\cdots,N$, in (1) [2]. Thus, we can conclude that time delays cause no particular difficulties in determining the stability of large scale systems with delays.

III. Concluding Remarks

The results in this note may be further applicable for the synthesis of decentralized stabilizing controller for large scale control system with time delays which was considered in [7].

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A Procedure to Eliminate Decentralized Fixed Modes with Reduced Information Exchange

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Abstract—**When decentralized fixed modes occur in a set of interconnected subsystems then complete pole assignment or stabilization (if they are unstable) is not possible. In this brief paper we present a way of choosing a new structure for the feedback matrix such that the fixed modes are eliminated and such that the exchange of information among subsystems is reduced. The fixed modes are characterized here by means of block diagonally dominant matrices. The procedure is illustrated on two examples.**

I. Introduction

The concept of fixed modes in decentralized control systems was introduced by Wang and Davison [1] and they correspond to the system eigenvalues that are not affected by decentralized feedback.

The fixed modes constitute a very important concept since they determine whether a system can be stabilized by decentralized control. The presence of unstable fixed modes prevents stabilization. Also, the presence of any fixed modes does not allow complete pole assignment.

The fixed modes also play an important role in determining whether a strongly connected system [2] can be made controllable and observable from a local controller and a local output.

Now, we know that the fixed modes are completely determined by the triple (C, A, B) and the structure of the feedback matrix. We call the fixed modes decentralized if they arise as a result of decentralized feedback.

In large scale systems, the decomposition into subsystems is often arbitrary. Clearly, a decomposition which yields fixed modes under decentralized feedback is not desirable. In this paper we present a constructive approach to changing the original decomposition in order to remove the fixed modes. This is done by changing the pattern of communication of information, i.e., by changing the structure of the feedback matrix.

The basic criteria used in defining the new decomposition is to remove fixed modes and, if possible, also reduce the information exchange.

The characterization of fixed modes is done here in terms of block diagonally dominant matrices [3]. The characterization provides a way of choosing a new feedback structure. Examples are also provided to illustrate the procedure.

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II. Characterizing the Decentralized Fixed Modes Through Block Diagonally Dominant Matrices

Consider the set of interconnected linear dynamical subsystems

$$\dot{x}_i = A_{ii}x_i + B_iu_i + \sum_{\substack{j=1 \\ j\neq i}}^{N} A_{ij}x_j \qquad i=1,\cdots,N \tag{1}$$

$$y_i = C_ix_i$$

where $x_i \in R^{n_i}$, $u_i \in R^{r_i}$, $y_i \in R^{m_i}$. The equations (1) can be written in matrix form as

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{2}$$

where

$$A = \{A_{ij}, i=1,\cdots,N, j=1,\cdots,N\} \in R^{n\times n}$$

$$B = \text{block diag}(B_1, B_2, \cdots, B_N) \in R^{n\times r}$$

$$C = \text{block diag}(C_1, C_2, \cdots, C_N) \in R^{m\times n}$$

$$n = \sum_{i=1}^{N} n_i \quad r = \sum_{i=1}^{N} r_i \quad m = \sum_{i=1}^{N} m_i.$$

The triple (C, A, B) in (2) is assumed to be controllable and observable.Consider also the set

$$\bar{K} = \{K \mid K = \text{block diag}(K_{11}, K_{22}, \cdots, K_{NN}), K_{ii} \in R^{r_i\times m_i}, i=1,\cdots,N\}.$$

By applying the decentralized output feedback law

$$u_i = K_{ii}y_i$$

we obtain the closed-loop system given by

$$\begin{aligned} \dot{x} &= (A + BKC)x \\ y &= Cx. \end{aligned} \tag{3}$$

The set of decentralized fixed modes of (C, A, B) is defined as

$$\Lambda(C, A, B, \bar{K}) = \bigcap_{K\in\bar{K}} \sigma[A + BKC] \tag{4}$$

where $\sigma[A + BKC]$ denotes the set of eigenvalues of $A + BKC$. Let the submatrices in the diagonal of $A + BKC$ be denoted by $\hat{A}_{ii}$, where

$$\hat{A}_{ii} = A_{ii} + B_iK_{ii}C_i \qquad i = 1\cdots N.$$

Then, $A + BKC$ can be represented as

$$A + BKC = \begin{bmatrix} \hat{A}_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & \hat{A}_{22} & \cdots A_{2N} & \\ \vdots & & & \vdots \\ A_{N1} & A_{N2} & \cdots & \hat{A}_{NN} \end{bmatrix}.$$

If the diagonal submatrices $\hat{A}_{ii}$ are nonsingular and if

$$\|\hat{A}_{ii}^{-1}\|^{-1} > \sum_{\substack{j=1 \\ j\neq i}}^{N} \|A_{ij}\| \qquad \text{for all } i = 1\cdots N$$

then $A + BKC$ is strictly block diagonally dominant.

The following theorem is taken from [3].

Theorem 1: If the matrix $A + BKC$ is strictly block diagonally dominant, then $A + BKC$ is nonsingular. The connection of this theorem with the decentralized fixed modes is made through the following corollary.

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