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# On Stabilization by Local State Feedback for Discrete-Time Large-Scale Systems with Delays in Interconnections

IL HONG SUH AND ZEUNGNAM BIEN

*Abstract*—**By employing an extended Nyquist array technique, a sufficient condition is obtained for decentralized stabilization of a class of discrete-time large-scale systems with delays in interconnections.**

## I. Introduction

Centralized control techniques have been known to be inefficient or even unsuccessful in some cases if applied for the control of large-scale dynamic systems [1]. To overcome the difficulties of centralized control methods, many researchers have proposed as alternatives various decentralized control methods involving simplification of model descriptions, effective procedures of testing the stability and/or hierachical optimization [1]. However, most of the decentralized control techniques developed so far are derived to handle the continuous-time systems [2], [3]. In particular, the assertion [2], [3] that there always exist local state feedback controllers stabilizing the large-scale systems in which delayed and/or nondelayed interactions occur only through the input of each controllable subsystem is true for the continuous dynamic systems, but may not directly apply for the discrete-time systems.

In this note it is shown that if, in addition to the controllability assumption on each subsystem as in the continuous-time case, certain restrictions are imposed on the interaction signal, then discrete-time large-scale systems can be stabilized by decentralized state feedback controllers. For this, an extended Nyquist array technique in [6] is employed.

It is noted that Chan and Desoer [3] utilized the Nyquist array technique by Rosenbrock [4] for the synthesis of decentralized stabilizing controller of the continuous-time large-scale systems. Thus, the result developed here may be viewed as an extension of Chan and Desoer's work [3] to the discrete time case.

Throughout the note, $\tilde{C}$ will denote the unit circle of the complex plane. $R^n$ denotes $n$-dimensional Euclidean vector space. For a given square matrix $A$, $A^{-1}$ and $\det A$ will denote the inverse and the determinant of $A$, respectively. For a given $n \times m$ matrix $B$ with entries $b_{ij}$ (that is, $B = [b_{ij}]_{i=n,\, j=m}$), $\|B\|_\infty$ will denote the norm defined [7] as

$$\|B\|_\infty = \max_{1 \le i \le n} \left\{ \beta_i \,|\, \beta_i = \sum_{j=1}^{m} |b_{ij}| \right\}.$$

## II. Synthesis of Decentralized Stabilizing Controller

Consider the large-scale system with $N$ subsystem $S_i$, $i = 1, 2, \cdots, N$, as shown in Fig. 1, where the dynamics of $S_i$ are given by

$$S_i:\ x_i(k+1) = A_i x_i(k) + b_i v_i(k), \qquad i = 1, 2, \cdots, N. \tag{1}$$

Here $x_i(k) \in R^{n_i}$ is the state vector, $v_i(t)$ is scaler input to the $i$th subsystem, and $A_i$ and $b_i$ are the $n_i \times n_i$ and $n_i \times 1$ constant matrices, respectively. Let each subsystem $S_i$ be interconnected to other systems by the following relation:

$$v_i(k) = u_i(k) + \sum_{\substack{j=1 \\ j \neq 1}}^{N} E_{ij} x_j(k - h_{ij}), \qquad i = 1, 2, \cdots, N. \tag{2}$$

Here $u_i(k)$ is the scaler control input to the $i$th subsystem $S_i$, $h_{ij} \geq 0$ is the delay-time in interconnections, and $E_{ij}$ is the $1 \times n_j$ constant matrix of the form

$$E_{ij} = \left[ e_{ij}^0 \ e_{ij}^1 \cdots e_{ij}^{n_j - 1} \right]. \tag{3}$$

Then from (1) and (2), the composite system can be represented as

$$S_i:\ x_i(k+1) = A_i x_i(k) + b_i u_i(k) + \sum_{\substack{j=1 \\ j \neq i}}^{N} b_i E_{ij} x_j(k - h_{ij}), \qquad i = 1, 2, \cdots, N. \tag{4}$$

For the continuous version of (4), it was shown in [2], [3] that under the assumption that each subsystem is controllable, there exist local decentralized state feedback controllers which stabilize the continuous-time large-scale systems. However, in the case of the discrete-time large-scale system in (4), local stabilizing state feedback controllers may fail to exist under the local controllability assumption only as in the continuous-time case. This fact is easily shown by a simple example in the following.

*Example 1:* In (4), let $N = 2$, $n_1 = n_2 = 1$, $A_1 = A_2 = 0$, $E_{12} = E_{21} = 10^2$, and $h_{12} = h_{21} = 0$. Let the local controller be $u_i(k) = -\alpha_i x_i(k)$, for each $i = 1, 2$. Then the characteristic equation of the closed-loop system is given by

$$d(z) = (z + \alpha_1)(z + \alpha_2) - 10^4. \tag{5}$$

It easily follows from (5) that both of the zeros of $d(z)$ lie in the unit circle $\tilde{C}$ of the $z$ plane only if

$$|\alpha_1 + \alpha_2| < 2, \quad \text{and} \quad |\alpha_1 \alpha_2 - 10^4| < 1. \tag{6}$$

However, there do not exist real numbers $\alpha_1$ and $\alpha_2$ satisfying (6), and hence the system cannot be stabilized by the local state feedback controls.

It will be shown that the discrete-time large-scale systems in (4) can be stabilized by local state feedback controller of the form

$$u_i(k) = F_i x_i(k), \qquad i = 1, 2, \cdots, N \tag{7}$$

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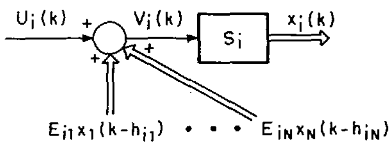


Fig. 1. The block diagram of the $i$th subsystem.

where $F_i$ is a $1\times n_i$ constant gain matrix given by

$$F_i=\left[-f_i^0\ -f_i^1\ \cdots\ -f_i^{n_i-1}\right], \tag{8}$$

if we impose, in addition to the assumption of local controllability, certain restrictions on the interaction signal of the system in (4). Specifically, it is assumed that the following conditions hold for the composite system in (4).

*Assumption A.1:* For each $i=1,2,\cdots,N$, $(A_i, b_i)$ is a controllable pair and, without loss of generality, is given in the controllable canonical form, i.e.,

$$A_i=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0\\ \vdots & \ddots & \ddots & \ddots & \vdots \\ & & & & 0\\ 0 & & \cdots & 0 & 1\\ -a_i^0 & -a_i^1 & \cdots & & -a_i^{n_i-1}\end{bmatrix}, \quad b_i=\begin{bmatrix}0\\ \vdots\\ 0\\ 1\end{bmatrix}. \tag{9}$$

*Assumption A.2:* For each $i=1,2,\cdots,N$, and each $j=1,2,\cdots,N$, let

$$\xi_{ij}\triangleq\|b_iE_{ij}\|_\infty=\sum_{k=0}^{n_j-1}|e_{ij}^k|. \tag{10}$$

Then there exist a set of positive scalers $d_i$, $i=1,2,\cdots,N$ such that

$$d_i^{-1}\sum_{\substack{j=1\\ j\neq i}}^{N} d_j\xi_{ji}<1, \qquad i=1,2,\cdots,N. \tag{11}$$

It is remarked that Assumption A.1 is typical in the sense that this assumption also appears in continuous-time cases, while Assumption A.2 is new and thus somewhat restrictive. Roughly speaking, Assumption A.2 may hold if all the norms of interconnection matrices $b_iE_{ij}$ in (10) are sufficiently small, which in turn implies that the interaction signals $(\Sigma_{j=1,\ j\neq i}^N b_iE_{ij}x_j(k-h_{ij}))$ in (4) are weak. The main result is now presented.

*Theorem 1:* Let the composite system in (4) satisfy Assumptions A.1 and A.2. Then the decentralized controllers in (7) stabilize the composite system in (4).

*Proof:* From (4) and (6), the characteristic equation $\boldsymbol{d}(z)$ of the closed-loop system can be written as

$$\boldsymbol{d}(z)=\det(zI-T(z)) \tag{12}$$

where

$$T(z)\triangleq\left[t_{ij}(z)\right]_{i=1,2,\cdots,N;\,j=1,2,\cdots,N}$$

is given by the block entries given by

$$t_{ij}(z)\triangleq\begin{cases}A_i+b_iF_i, & i=j,\\ b_iE_{ij}z^{-h_{ij}}, & i\neq j.\end{cases} \tag{13}$$

Since $T(z)$ in (13) is in a generalized companion form, it is obvious [5] that there exists $N\times N$ polynomial matrix

$$\boldsymbol{P}(z)\triangleq\left[P_{ij}(z)\right]_{i=1,2,\cdots,N;\,j=1,2,\cdots,N}$$

such that

$$\boldsymbol{d}(z)=\det \boldsymbol{P}(z) \tag{14}$$

where

$$p_{ij}(z)=\begin{cases} z^{n_i}+\sum_{k=0}^{n_i-1}\left(a_i^k+f_i^k\right)z^k, & i=j\\ z^{-h_{ij}}\sum_{k=0}^{n_j-1}e_{ij}^kz^k, & i\neq j.\end{cases} \tag{15}$$

Thus, it suffices to show the existence of $f_i^k$ for each $i=1,2,\cdots,N$ and $k=0,1,\cdots,n_i-1$ such that all the zeros of $\det \boldsymbol{P}(z)$ lie in the unit circle of the $z$ plane.

For this, suppose that for each $z\in\bar{C}$, there exists a positive scalar $d_i$ such that

$$d_i|P_{ii}(z)|>\sum_{\substack{j=1\\ j\neq 1}}^{N} d_j|P_{ji}(z)|, \qquad i=1,2,\cdots,N. \tag{16}$$

Then $R(z)$ defined by

$$R(z)\triangleq\boldsymbol{P}(z)\operatorname{diag}\{1/z^{n_i}\}, \qquad i=1,2,\cdots,N \tag{17}$$

is a Hadamard matrix on $\bar{C}$ [6]. Also it follows from (17) that all the poles of $\det R(z)$ lie in $\bar{C}$ of the $z$ plane and each diagonal element of $R(z)$ has its poles in $\bar{C}$ of the $z$ plane. Thus if all the zeros of $P_{ii}(z)=0$ lie in $\bar{C}$ of the $z$ plane for all $i=1,2,\cdots,N$, then by the Nyquist array theorem in [6], $\det \boldsymbol{P}(z)$ has no zeros outside of $\bar{C}$ in the $z$ plane. From this, our remaining task is to show the existence of $f_i^k$ for each $i=1,2,\cdots,N$ and $k=0,1,\cdots,n_i-1$ such that

1) $d_i|P_{ii}(z)|>\sum_{\substack{j=1\\ j\neq i}}^{N} d_j|P_{ji}(z)|$, for all $z\in\bar{C}$, and

2) $P_{ii}(z)$ has no zeros outside of $\bar{C}$ in the $z$ plane.

To find such a set of gain parameters, let, for each $j=1,2,\cdots,N$, $F_j$ be chosen such that all the eigenvalues of $(A_j+b_jF_j)$ are zero. Then for each $z\in\bar{C}$ and $i=1,2,\cdots,N$,

$$|P_{ii}(z)|=|z^{n_i}|=1 \tag{18}$$

and thus from Assumption 2

$$d_i|P_{ii}(z)|=d_i>\sum_{\substack{j=1\\ j\neq i}}^{N} d_j\xi_{ji} \tag{19}$$

$$=\sum_{\substack{j=1\\ j\neq i}}^{N} d_j\left(\sum_{k=0}^{n_i-1}|e_{ji}^kz^k|\right)|z^{-h_{ji}}| \tag{20}$$

$$>\sum_{\substack{j=1\\ j\neq i}}^{N} d_j\left|\sum_{k=0}^{n_i-1}e_{ji}^kz^k\right|\left|z^{-h_{ji}}\right| \tag{21}$$

$$=\sum_{\substack{j=1\\ j\neq i}}^{N} d_j|P_{ji}(z)|. \tag{22}$$

This completes the proof.

## III. Concluding Remarks

A sufficient condition for decentralized stabilization of a class of discrete-time large-scale systems with delays in interconnections was obtained by employing an extended Nyquist array technique [6]. The result derived may apply if each subsystem is controllable and the interactions between subsystems are small. It is noted that in the design of computer-based local controllers for continuous-time large-scale systems, the sampling time can be chosen to be sufficiently small to make the resulting discrete-time systems always satisfy Assumption A.2. In the vector Lyapunov function method in [8], an assumption similar to Assumption A.2 is used for the case when $n_i$ is equal to unity for $i = 1,2,\cdots,N$. It is further remarked that in the proof of Theorem 1 the interactions with delays are handled without any increase of the dimensions of subsystems, while those delay terms are not easily dealt with by other approaches [8], [9].

Finally, it is observed that there exists a set of numbers $\{d_1, d_2, \cdots, d_N\}$ such that (11) holds if and only if all the leading principal minors of $I - [\xi_{ij}]_{i=1,\cdots,N\ j=1,\cdots,N}$ are positive, which may be utilized to test Assumption A.2.

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# A Note on "Multilayer Control of Large Markov Chains"

A. HAURIE

*Abstract* —**It is shown that the multilayer control scheme of the above paper[1] can be constructed by using available results on Markov renewal theory and semi-Markov decision processes.**

## I. INTRODUCTION

In the above paper[1] Forestier and Varaiya have investigated a two-layer feedback control structure for the control of a plant modeled as a Markov chain with a large number of states. This paper was related to the important literature in control theory dealing with large-scale systems where different parts of the system under study operate at "different time scales."

The aim of this note is to show that a slightly different interpretation could be given to a two-layer feedback control of a Markov chain. When a Markov decision process possesses a very large state set $S$, it is possible to associate with any proper subset $B$ of $S$ a semi-Markov decision process *defined on the same time scale as the original process*, and which constitutes an aggregation of it.

A multilayer control scheme can thus be constructed by using many available results on Markov renewal theory and semi-Markov decision processes. The proposition in the next section gives an alternate and quick proof of the main results of Forestier and Varaiya.

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## II. THE MULTILAYER CONTROL SCHEME: A SEMI-MARKOV DECISION PROCESS RESULTING FROM THE AGGREGATION OF A MARKOV DECISION PROCESS

With the notations of the paper[1] controlled process is denoted $s_t$, $t = 0,1,\cdots$ with values in $S \triangleq \{1,\cdots,s\}$. If $s_t = i$, the control $u_t$ can be chosen in a prescribed set $U(i)$. A stationary strategy is an element $u = (u(1),\cdots,u(s)) \in U \triangleq U(1)\times U(2)\times \cdots \times U(s)$. For each $u$ the process $s_t$ is a Markov chain with stationary transition probability matrix $P(u) \triangleq \{P_{ij}(u)\}$ where

$$P_{ij}(u) = P_{ij}(u(i)) \triangleq \Pr[s_{t+1} = j | s_t = i, u_t = u(i)]. \tag{1}$$

There is a cost $k(i, u(i))$ associated with the process being in state $i$ and the control being $u(i)$. Under the *strong ergodicity assumption* (≪ for each $u$ the chain $s_t$ has a single ergodic class consisting of all the states ≫), one can associate with the stationary strategy $u$ the long-term mean average cost

$$J(u) \triangleq \lim_{T\to\infty} \frac{1}{T+1} E\left[\sum_{t=0}^{T} k(s_t, u(s_t))\right]. \tag{2}$$

The determination of an optimal strategy $u$ minimizing the cost (2) is a classical problem fully treated in the operations research literature (see [2]). Numerical algorithms permitting the computation of optimal strategies for large-scale Markov chains have recently been proposed, although many practical problems remain out of reach of these theories because of the sheer size of the state set $S$ and action sets $U(i)$, $i \in S$. In such situations an aggregation technique must be used. The technique proposed by Forestier and Varaiya is based on the following.

1) The restriction of the state $S$ to a proper subset $B \triangleq \{1,2,\cdots,b\} \subset S$ called the set of *boundary states*;

2) the consideration, for each state $\beta$ in $B$ of a subset $V^\beta$, of the set $U$ of possible strategies for $s_t$.

Each time a boundary state $\beta$ in $B$ is reached, a particular strategy $v^\beta$ is picked in $V^\beta$ and the evolution of $s_t$ is governed by the transition probability matrix $P(v^\beta)$ until the next random time at which another state $\beta'$ in $B$ will be reached, etc.

Let us define the supervisor process $b_t$, $t = 0,1,\cdots$ with values in $B$ as follows. We assume that $s_0 = b_0 \in B$ and let $0 \equiv T_0 < T_1 < \cdots < T_n$ be the random times at which $s_t$ is in $B$, i.e.,

$$T_{n+1}(\omega) = \min\{t > T_n(\omega); s_t(\omega) \in B\}. \tag{3}$$

Here $\omega$ denotes the sample path. The supervisor process is

$$b_t(\omega) = s_{T_n(\omega)}(\omega) \qquad T_n(\omega) \leq t < T_{n+1}(\omega). \tag{4}$$

Notice that the definition of the supervisor process $b_t$ is given here in the same time scale as the lower layer process $s_t$. This differs slightly from the definition of the supervisor process given by Forestier and Varaiya. We can state the following.

*Lemma 1:* ≪Given a supervisor strategy $v = (v^1, v^2,\cdots,v^b) \in V^1 \times \cdots \times V^b$, the supervisor process $b_t$ defined by (3) and (4) is a semi-Markov process with a strongly ergodic embedded Markov chain.≫

*Proof:* Direct consequence of the strong ergodicity assumption on $s_t$. The supervisor process is, in fact, a particular case of a Markov renewal

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