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Brief Paper

Analysis and Design of Two Types of Digital Repetitive Control Systems*

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Key Words—Linear digital repetitive control system; gain adjusting; higher-order repetitive function; relative error transfer function.

Abstract—Two types of linear digital repetitive control systems are designed and analyzed to reduce the error spectrum, including not only harmonic but also non-harmonic components. First, a novel gain adjusting algorithm is suggested for conventional and modified repetitive control systems with modelling uncertainties, in which the gain of the repetitive controller is adjusted to reduce the infinite norm of error in the frequency domain. For this, the relative error transfer function is newly defined as the ratio of the error spectrum for the system with a repetitive controller to the error spectrum for the system without a repetitive controller. Secondly, as an alternative to a repetitive control system with the gain adjusting, a repetitive control system with higher-order repetitive function is analyzed and designed, where the weightings of the higher-order repetitive function are determined in such a way that the infinite norm of the relative error transfer function is minimized. To show the validity of the proposed methods, computer simulation results are illustrated for a typical disk-drive head-positioning servo system.

1. *Introduction*

Many repetitive control systems, based on the internal model principle, have been proposed to effectively reduce repetitive and/or harmonic errors with a known period in various control systems performing repetitive tasks, such as industrial robot control systems (Tsai *et al.*, 1988), disk-drive head-positioning servo systems (Chew and Tomizuka, 1989, 1990) and capstan motor speed control systems in camcorders (Gotou *et al.*, 1991). Repetitive control systems were originally studied in the continuous-time domain (Inoue *et al.*, 1981; Hara *et al.*, 1988). Digital repetitive controllers were then proposed by many researchers (see e.g. Tomizuka *et al.*, 1988; Chew and Tomizuka, 1989, 1990). Several modified repetitive controllers were also proposed to obtain robustness under modelling uncertainties (Hara *et al.*, 1988; Chew and Tomizuka, 1989, 1990). Unfortunately, when such controllers are employed to reduce harmonic errors, non-harmonic error components are often amplified. To alleviate this diffculty, Inoue (1990) showed that the gain reduction of repetitive controllers or the application of a smoothing function with equal weightings over periods, which will be hereinafter called a higher-order repetitive function, could be possible solutions in the design of repetitive controllers. However, it has not been discussed when and how much the gain of repetitive controllers should be reduced and/or adjusted. Furthermore, Inoue's gain adjusting method could not be applied to practical repetitive control systems with modelling uncertainties, since it did not consider these uncertainties. Also, his higher-order repetitive function with equal weightings seems to be ineffective.

In this paper, two types of linear digital repetitive control systems are designed and analyzed to reduce the error spectrum, including not only harmonic but also non-harmonic components. First, a novel gain adjusting algorithm is suggested for conventional and modified repetitive control systems with modelling uncertainties, in which the gain of the repetitive controller is adjusted to reduce the infinite norm of error in the frequency domain. For this, the relative error transfer function is newly defined as the ratio of the error spectrum for the system with a repetitive controller to the error spectrum for the system without a repetitive controller. Secondly, as an alternative to a repetitive control system with gain adjusting, a repetitive control system with higher-order repetitive function is designed and analyzed, where, instead of equal weightings (Inoue, 1990), weightings of the higher-order repetitive function are determined in such a way that the infinite norm of the relative error transfer function is minimized.

To show the validity of the proposed methods, computer simulation results are illustrated for a typical disk-drive head-positioning servo system.

2. *A gain adjusting algorithm for conventional and modified repetitive control systems*

Consider the linear SISO digital conventional repetitive control system shown in Fig. 1(a), where $R(z)$ is the z-transform of the reference input, $C(z)$ the z-transform of the controlled output, $E(z)$ the z-transform of the error, $G_r(z)$ the transfer function of the repetitive controller, $G_o(z)$ the transfer function of the controlled system, and $G_i(z)$ a stable rational function of z. $G_i(z)$ is often chosen as the inverse transfer function of $G_{so}(z) \equiv G_o(z)/[1 + G_o(z)]$, if perfect identification of the controlled system is available. K_r, z^{-N}, T and NT are respectively the gain of the repetitive controller, the dead-time element, the sampling time, and the dead-time length, which is chosen to be equal to the period of the fundamental component of the repetitive errors. A signal will be called harmonic if it is periodic and has no frequency component other than $2\pi k/NT$ (in rad s^{-1}) for

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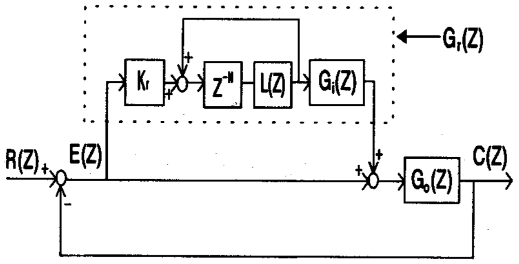


Fig. 1. Block diagram representing three repetitive control systems: (a) a conventional repetitive control system ($L(z)=1$); (b) a modified repetitive control system ($L(z)=q(z)$); (c) a repetitive control system with a higher-order repetitive function ($L(z)=W_j(z)$).

$k \in I$, where I denotes the set of integers. Otherwise, the signal will be called non-harmonic.

Let $G_{so}(z)$ be given by

$$G_{so}(z)=\frac{B(z)}{A(z)}=\frac{B^+(z)B^-(z)}{A(z)}, \tag{1}$$

where $A(z)$ and $B(z)$ are respectively nth and mth-order polynomials in z^{-1}. In (1), $B^+(z)$ and $B^-(z)$ are respectively the cancellable and uncancellable parts of $B(z)$. That is, $B^-(z)$ and $B^+(z)$ comprise roots of $B(z)$ on or outside the closed unit circle and inside the closed unit circle respectively. Let $G_i(z)$ be modelled as

$$G_i(z)=\frac{A(z)B^-(z^{-1})}{B^+(z)b}, \tag{2}$$

where b is a constant satisfying the inequality given by (Tomizuka *et al.*, 1988)

$$b \geq \max_{\omega\in[0,\pi]} |B^-(e^{j\omega})|^2.$$

In (2), $B^-(z^{-1})$ can be obtained by replacing every z in $B^-(z)$ with z^{-1}. Also let $G_{cl}(z)$ be the closed-loop transfer function of the whole system in Fig. 1(a). Then $G_{cl}(z)$ can be obtained as

$$\frac{C(z)}{R(z)}\equiv G_{cl}(z)=\frac{[1+G_r(z)]G_o(z)}{1+[1+G_r(z)]G_o(z)}$$
$$=\frac{[1-z^{-N}+K_r z^{-N}G_i(z)]G_{so}(z)}{1-z^{-N}+K_r z^{-N}G_i(z)G_{so}(z)}. \tag{3}$$

Further, let $G_e(z)$ be the error transfer function, defined as $G_e(z)\equiv 1-G_{cl}(z)$. Then it can be shown that $G_e(z)$ can be rewritten in the product form $G_e(z)\equiv G_{re}(z)G_{eo}(z)$, where $G_{eo}(z)\equiv 1-G_{so}(z)$ corresponds to the error transfer function when the repetitive controller $G_r(z)$ is not employed and $G_{re}(z)\equiv[1+G_r(z)G_s(z)]^{-1}$ can be considered as the relative error transfer function defined as the ratio of $G_e(z)$ to $G_{eo}(z)$. Thus we shall call the spectrum of $G_{re}(z)$ the '*relative error transfer function*'. It is notable that when $K_r=1$ in (3), $G_{re}(z)$ becomes the *frequency response of the error to the disturbance*, as proposed by Inoue (1990). In this sense, $G_{re}(z)$ can be considered as a generalized version of the *frequency response of the error to the disturbance*.

It is well known that if the repetitive control system $G_{cl}(z)$ in (3) is asymptotically stable then harmonic error components can be eliminated by digital repetitive control action (Tomizuka *et al.*, 1988). But unfortunately, when such a repetitive controller is employed to reduce harmonic errors, non-harmonic error components are often amplified. To alleviate this problem, Inoue (1990) showed that the gain reduction of a repetitive controller could be a possible solution in the design of a repetitive controller. However, it has not been discussed when and by how much the gain should be reduced and/or adjusted. Therefore we shall propose a gain adjusting algorithm for a repetitive controller to simultaneously reduce harmonic and non-harmonic error components, where the gain of a repetitive controller is adjusted to reduce the infinite norm of $E(z)$. The adjustment method is established on the basis of a quantitative analysis of the relative error transfer function.

To be specific, let $E_o(z)$ denote the error for the system without a repetitive controller, which is given by $E_o(z)=G_{eo}(z)R(z)$. Since $E(z)=G_e(z)R(z)$ and $G_{re}(z)=G_e(z)/G_{eo}(z)$, the relation between $E(z)$ and $E_o(z)$ can be obtained as $E(z)=G_{re}(z)E_o(z)$. Let $z=e^{j\omega T}$ and $\theta=\omega NT$. Then $G_{re}(z)$ becomes a function of K_r and θ, and can be rewritten as

$$E(K_r,\theta)=G_{re}(K_r,\theta)E_o(\theta). \tag{4}$$

Let $|E|_m=\max_{0\leq\theta\leq 2\pi}|E(\theta)|$ and $\theta_m=\arg[\max_{0\leq\theta\leq 2\pi}|E(\theta)|]$, respectively the infinite norm of $E(\theta)$ and the frequency at which $|E|_m$ can be found. Also let $\nabla_{K_r}(\cdot)$ denote the partial derivative of $(\cdot)$ with respect to K_r. Then $|E|_m$ can be seen to be reduced by adjusting the gain K_r as follows.

Observation 2.1. Assume that the repetitive control system given by (3) is asymptotically stable. Then $|E|_m$ is reduced if K_r is increased when $\nabla_{K_r}(|G_{re}(K_r,\theta_m)|)<0$ and if K_r is decreased when $\nabla_{K_r}(|G_{re}(K_r,\theta_m)|)>0$.

Proof. From (4), the magnitude of $E(K_r,\theta)$ for $\theta=\theta_m$ can be written as $|E(K_r,\theta_m)|=|G_{re}(K_r,\theta_m)|\,|E_o(\theta_m)|=|E|_m$. Thus the differential of $|E(K_r,\theta_m)|$ with respect to K_r can be obtained as

$$\delta(|E(K_r,\theta_m)|)=\frac{\partial(|G_{re}(K_r,\theta_m)|)}{\partial(K_r)}\delta(K_r)\,|E_o(\theta_m)|. \tag{5}$$

From (5), we know that $\delta(|E(K_r,\theta_m)|)<0$ if $\delta(K_r)>0$ and $\nabla_{K_r}(|G_{re}(K_r,\theta_m)|)<0$ or if $\delta(K_r)<0$ and $\nabla_{K_r}(|G_{re}(K_r,\theta_m)|)>0$. This completes the proof. □

It can be seen from Observation 2.1 that θ_m and the sign of $\nabla_{K_r}(|G_{re}(K_r,\theta_m)|)$ should be determined to reduce $|E|_m$. θ_m can be obtained in real time by conventional spectral analysis techniques such as the short-time fast Fourier transform or filter banks (Rabiner and Schafer, 1978). The sign of $\nabla_{K_r}(|G_{re}(K_r,\theta_m)|)$ can also be easily determined on the basis of a quantitative analysis of $G_{re}(K_r,\theta_m)$. Now, to consider the modelling uncertainties, let the unmodelled dynamics $\Delta(z)$ be represented in the form of a multiplicative modelling error. Then the relation between the true system transfer function $G_s(z)$, and the nominal system transfer function $G_{so}(z)$ can be written as

$$G_s(z)=G_{so}(z)[1+\Delta(z)]. \tag{6}$$

Let $\Delta_R(\theta)$ and $\Delta_I(\theta)$ be the real and the imaginary parts of $\Delta(z)$ when $z=e^{j\omega T}$ and $\theta=\omega NT$. Also let $|\Delta_R(\theta)|\leq\delta$, and $|\Delta_I(\theta)|\leq\delta$. Assume that $G_{so}(z)$ has no uncancellable zeros. Then the relation between $G_i(z)$ in (2) and $G_s(z)$ in (6) is given by

$$G_i(z)G_s(z)=1+\Delta(z), \tag{7}$$

and thus $G_{re}(K_r,\theta)$ can be obtained as

$$G_{re}(K_r,\theta)=\frac{1-e^{-j\theta}}{1-e^{-j\theta}+K_r e^{-j\theta}[1+\Delta(\theta)]}. \tag{8}$$

Then $|G_{re}(K_r,\theta)|$ can be computed as

$$|G_{re}(K_r,\theta)|=\left[\frac{\alpha_1}{K_r^2(1+\beta_1)-K_r(\alpha_1+\gamma_1)+\alpha_1}\right]^{1/2}, \tag{9}$$

where

$$\alpha_1=2(1-\cos\theta),\quad \beta_1=\Delta_R(\theta)^2+\Delta_I(\theta)^2+2\Delta_R(\theta),$$
$$\gamma_1=2[\Delta_R(\theta)-\sin\theta\,\Delta_I(\theta)-\cos\theta\,\Delta_R(\theta)].$$

Since $|G_{re}(K_r,\theta)|$ in (9) is an even periodic function of the fundamental harmonic frequency $\omega_0=2\pi/NT$, $B_k=\{\omega \mid \omega_0 k\leq\omega\leq\omega_0(k+1)\}$ for any positive integer k can be mapped on the normalized frequency region given as $\{\theta \mid 0\leq\theta\leq 2\pi\}$. Thus we can show that the frequencies at

$\theta = 0$ or 2π correspond to the harmonic frequencies, and all the other frequencies except $\theta = 0$ and 2π correspond to the non-harmonic frequencies. Now, to find the sign of $\nabla_{K_r}(|G_{re}(K_r, \theta_m)|)$, let $K_{ro} \in (0, 2)$ be defined as $K_{ro} \equiv \arg(\max_{0<K_r<2} |E(K_r, \theta)|)$ for fixed θ. Then K_{ro} can be determined as $K_{ro} = (\alpha_1 + \gamma_1)/[2(1 + \beta_1)]$, by solving the equation given by $\nabla_{K_r}(|G_{re}(K_r, \theta)|) = 0$. Since $|\Delta_R(\theta)| \le \delta$ and $|\Delta_I(\theta)| \le \delta$, as in (6), K_{ro} also satisfies the inequalities

$$K_1 \le K_{ro} \le K_2, \tag{10}$$

where

$$K_1 = \frac{(1-\delta)(1-\cos\theta) - \delta\,|\sin\theta|}{1+\delta^2},$$

$$K_2 = \frac{(1-\cos\theta) + \delta\,|\sin\theta|}{(1-\delta)^2}.$$

It can be shown from (10) that if $K_r < K_1$ then $\nabla_{K_r}(|G_{re}(K_r, \theta)|) > 0$, while if $K_r > K_2$ then $\nabla_{K_r}(|G_{re}(K_r, \theta)|) < 0$ for fixed θ. Note that if θ_m is known, K_1 and K_2 can be computed using (10). Thus, by comparing the currently employed gain K_r with K_1 and K_2, the sign of $\nabla_{K_r}(|G_{re}(K_r, \theta)|)$ can easily be found. However, if $K_1 \le K_r \le K_2$, it is difficult to determine the sign of $\nabla_{K_r}(|G_{re}(K_r, \theta)|$. In this case, gain adjusting should not be performed. When $K_r \le 0$ or $K_r \ge 2$, gain adjusting should also not be performed, since the system can be stable when $0 < K_r < 2$ (Tomizuka *et al.*, 1988). Note that if perfect modelling of $G_s(z)$ is available then K_{ro} can be simply obtained as $K_{ro} = K_1 = K_2 = (1 - \cos\theta)$.

To show non-harmonic behavior effectively under the perfect modelling assumption, magnitude plots of $G_{re}(K_r, \theta)$ for various fixed K_r and various fixed θ are sketched in Fig. 2. It can be seen that the maximum magnitude of $G_{re}(K_r, \theta)$, denoted by $|G_{re}(K_r, \theta)|_m$, decreases monototically as K_r decreases monototically. This can also be shown by computing $|G_{re}(K_r, \theta)|_m$, given as $|G_{re}(K_r, \theta)|_m = 2/(2 - K_r)$ at $\theta = \pi$.

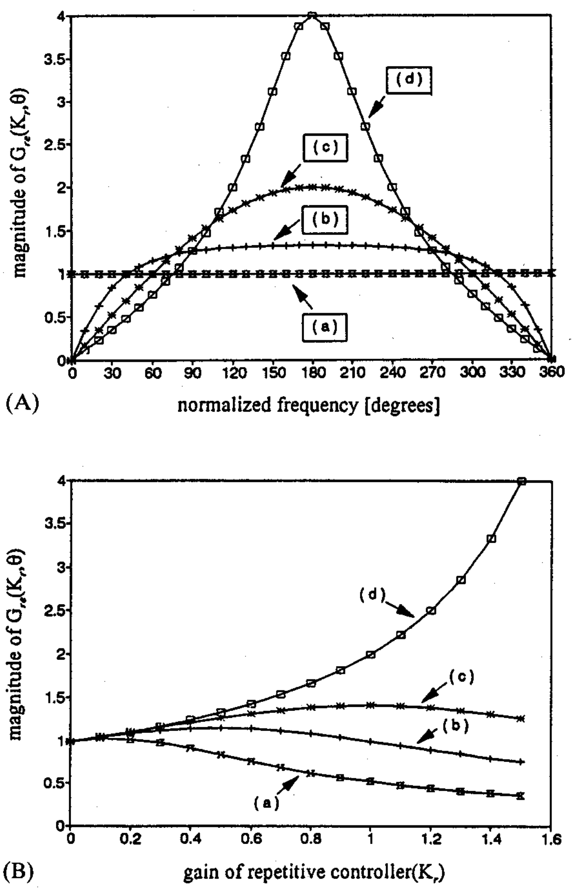


Fig. 2. (A) Magnitude plots of $G_{re}(K_r, \theta)$ for $K_r = 0$ (a), 0.5 (b), 1.0 (c) and 1.5 (d). (B) Magnitude plots of $G_{re}(K_r, \theta)$ for $\theta = 30°$ (a), 60° (b), 90° (c) and 180° (d).

Thus K_r should be adjusted to be as small as possible if non-harmonic frequency occurs only at $\theta = \pi$ and $|G_{re}(K_r, \theta)|_m$ is to be reduced. The gain reduction technique suggested by Inoue (1990) has been developed on the basis of this observation, but the reduction of K_r should lead to a deterioration in the error convergence rate. This trade-off makes lead to a difficulty in choosing K_r. Furthermore, if $\theta \ne \pi$, as is often the case in practice, the gain reduction technique cannot guarantee the decrease of $|G_{re}(\theta)|$. In fact, in the case of a disk-drive head-positioning servo system (Chew and Tomizuka, 1989, 1990), since non-harmonic frequency components of $|E(\theta)|$ are often included together with a few dominant non-harmonic frequencies in the reference signal due to the spindle bearings (Bouchard *et al.*, 1987) and white noise, the simple gain reduction technique cannot be applied in practice.

On the basis of the above quantitative analysis of $G_{re}(K_r, \theta)$, we shall propose a gain adjusting algorithm to reduce the infinite norm of the error in the frequency domain as follows.

Algorithm I. For the repetitive control system given by (3), determine the correctional gain amount $\Delta K_r(1) > 0$ to be used at first iteration. Let i be denoted as the iteration number. Also let $|E|_m(i)$, $K_r(i)$, $\theta_m(i)$, $\Delta K_r(i)$, $K_1(i)$ and $K_2(i)$ be $|E|_m$, K_r, θ_m, ΔK_r, K_1 and K_2 at the ith iteration. For initialization, let $i = 1$ and $|E|_m(0) = 0$. Let $K_r(1) = 1$ for a fast error convergence rate. To reduce steady-state errors or fluctuations of $|E|_m(i)$ and $K_r(i)$, $\Delta K_r(i)$ should be diminished, provided that $|E|_m(i)$ is sufficiently small. For this, if $|E|_m(i) > |E|_m(i-1)$ or $\theta_m(i) \ne \theta_m(i-1)$, ΔK_r is to be decreased. Here, the diminishing factor of $\Delta K_r(i)$ will be chosen as $\frac{1}{2}$. Then we apply the following steps:

step 1: find $|E|_m(i)$ and $\theta_m(i)$ by a spectal analysis of $E(i)$;
step 2: $\Delta K_r(i) = \frac{1}{2}\Delta K_r(i-1)$ if $|E|_m(i) - |E|_m(i-1) > 0$ or $\theta_m(i) \ne \theta_m(i-1)$;
step 3: compute $K_1(i)$ and $K_2(i)$ by substituting $\theta_m(i)$ into (10);
step 4: update $K_r(i+1)$ such that

$$K_r(i+1) = \begin{cases} K_r(i) - \Delta K_r(i) & (K_r(i) - K_1(i) < 0), \\ K_r(i) + \Delta K_r(i) & (K_r(i) - K_2(i) > 0), \\ K_r(i) & (K_1(i) \le K_r(i) \le K_2(i)); \end{cases}$$

step 5: update the iteration number as $i = i + 1$;
step 6: go to step 1.

Instead of conventional repetitive control, several modified repetitive control systems have been proposed (Hara *et al.*, 1988; Chew and Tomizuka, 1989, 1990) to obtain robustness of the repetitive control systems under modelling uncertainties, in which low-pass filtering is employed as shown in Fig. 1(b). To be specific, let $q(z)$ be the FIR low-pass filter transfer function. Then the closed-loop transfer function of the modified repetitive control system, denoted by $G_{clm}(z)$ can be obtained by simply replacing z^{-N} with $z^{-N}q(z)$ in (3). Note that stable region of K_r can be relatively enlarged by introducing $q(z)$ (Chew and Tomizuka 1989, 1990).

In the modified repetitive control system, if the relation between $G_s(z)$ and $G_i(z)$ can be represented as (7) under the modelling uncertainties, the relative error transfer function $G_{rem}(z)$ can be rewritten as

$$G_{rem}(z) = \frac{1 - z^{-N}q(z)}{1 - z^{-N}q(z) + K_r z^{-N}q(z)[1 + \Delta(z)]}. \tag{11}$$

To be specific, consider the FIR low-pass filter in Fig. 1(b), which satisfies the conditions given by $0 < q(z) \le 1$, and $\angle(q(z)) = 0$.

Theorem 2.1. Assume that the modified repetitive control system $G_{clm}(z)$ is asymptotically stable, and $\mathrm{Re}\,[1 + \Delta(z)] > 0$ at harmonic frequencies. Then harmonic errors are asymptotically decreased.

Proof. $|G_{rem}(z)|$ at harmonic frequencies can be obtained as

$$|G_{rem}(z)|_{z=\exp(j\omega_0 kT)} = \left|\frac{1-|q(z)|}{1-|q(z)|+K_r|q(z)|(1+\Delta(z))}\right|_{z=\exp(j\omega_0 kT)}$$

Since $0<|q(z)|\le 1$ and $\mathrm{Re}[1+\Delta(z)]>0$ at harmonic frequencies, $|G_{re}|<1$ at harmonic frequencies. This completes the proof. □

Note that when the unmodelled dynamics is given as a simple first-order ignored dynamics, $1+\Delta(z)=q_0/(z+p_0)$, where

$$q_0>0,\quad |p_0|\ll 1,\quad k\ll N,\quad \mathrm{Re}[1+\Delta(z)]|_{z=\exp(j\omega_0 kT)}>0$$

holds, because

$$\mathrm{Re}[1+\Delta(z)]|_{z=\exp(j\omega_0 kT)} = \mathrm{Re}[1+\Delta(z)]|_{z=\exp(j2\pi k/N)}$$

$$= \mathrm{Re}\left[\frac{q_0}{\exp(j2\pi k/N)+p_0}\right] = q_0\left[\cos\left(\frac{2\pi}{N}k\right)+p_0\right]>0$$

if $q_0>0$, $|p_0|\ll 1$ and $k\ll N$. The condition that k (order of harmonics) $\ll N$ (deadtime element NT divided by sampling time T), is reasonable in practice, since the dominant harmonic errors are usually composed of low-order harmonic components in many realistic cases (Chew and Tomizuka, 1989, 1990), and high-order harmonics are usually filtered out through the controlled dynamics with low-pass filtering characteristics.

In the case of modified repetitive control, $|G_{rem}(z)|$ at harmonic frequencies is no longer zero when $|q(z)|\ne 1$. And $|q(z)|$ becomes smaller as the frequency is increased owing to low-pass filtering characteristics, which makes $|G_{rem}(z)|$ of the modified repetitive control system larger than $|G_{re}(z)|$ of the conventional repetitive control system at harmonic frequencies. This implies that low-pass filtering usually degrades the harmonic error rejection performances of repetitive control actions. Now we shall show that gain adjusting algorithm similar to that for the repetitive control system in (3) can be established for the modified repetitive control system. For this, let $z=e^{j\omega T}$, $\theta=\omega NT$ and $|q(z)|=q(\theta)$. Then (11) can be written as

$$G_{rem}(K_r,\theta)=\frac{1-e^{-j\theta}q(\theta)}{1-e^{-j\theta}q(\theta)+K_r e^{-j\theta}q(\theta)[1+\Delta(\theta)]}, \quad (12)$$

and $|G_{rem}(K_r,\theta)|$ can be obtained as

$$|G_{rem}(K_r,\theta)|=\left[\frac{\alpha_2}{K_r^2 q(\theta)^2(1+\beta_2)-K_r\gamma_2+\alpha_1}\right]^{1/2}, \quad (13)$$

where

$$\alpha_2=1+q(\theta)^2-2q(\theta)\cos\theta,\quad \beta_2=\Delta_R(\theta)^2+\Delta_I(\theta)^2+2\Delta_R(\theta),$$

$$\gamma_2=2q(\theta)^2[1+\Delta_R(\theta)]-2q(\theta)[\cos\theta+\sin\theta\,\Delta_I(\theta)+\cos\theta\,\Delta_R(\theta)].$$

By solving the equation given by $\nabla_{K_r}(|G_{rem}(K_r,\theta)|)=0$, K_{ro} can be obtained as

$$K_{ro}=\frac{\gamma_2}{2q(\theta)^2(1+\beta_2)}.$$

Since $|\Delta_R(z)|\le\delta$ and $|\Delta_I(z)|\le\delta$, K_{ro} also satisfies the inequalities

$$K_{m1}\le K_{ro}\le K_{m2}, \quad (14)$$

where

$$K_{m1}=\frac{(1-\delta)[1-\cos\theta/q(\theta)]-\delta|\sin\theta|/q(\theta)}{1+\delta^2},$$

$$K_{m2}=\frac{1-\cos\theta/q(\theta)+\delta|\sin\theta|/q(\theta)}{(1-\delta)^2}.$$

Thus, by simply replacing (10) with (14) in Algorithm I, we can show that $|E|_m$ can be effectively reduced by employing our proposed gain adjusting algorithm. If perfect modelling is available, K_{ro} can be obtained as $K_{ro}=K_{m1}=K_{m2}=1-\cos\theta/q(\theta)$. Note here that in the case of the conventional repetitive control system in (3), $|E(\theta)|$ is zero at $\theta=0$ or 2π regardless of $K_r\in(0,2)$, and thus harmonic errors are not related to the gain adjusting. However, in the case of the modified repetitive control system, $|E(\theta)|$ is no longer zero at $\theta=0$ or 2π when $q(z)\ne 1$. Furthermore, a simple gain reduction technique cannot be applied, since $|E(\theta)|$ may increase as K_r decreases. In this sense, our proposed gain adjusting algorithm can play a critical role in reducing $|E(\theta)|_m$ for the modified repetitive control system rather than for the conventional repetitive control system in (3).

2.1. *Numerical examples.* Consider a typical disk-drive head-positioning servo system, where input–output transfer function $G_{so}(z)$ in (1) is given by

$$G_{so}(z)=\frac{5.01-14.0z^{-1}+14.2z^{-2}-6.0z^{-3}+0.88z^{-4}}{1-0.874z^{-1}-0.992z^{-2}+0.882z^{-3}}.$$

Here three cases of modelling uncertainty are to be investigated:

(i) perfect modelling;
(ii) with the maximum unmodelled portions of $G_s(z)$, defined as $\sqrt{2}\delta$, given as 0.2;
(iii) with $\sqrt{2}\delta$ given as 0.4.

The sampling time T and number of sectors of the hard-disk drive system N giving the dead-time length are chosen as 410 μs and 41 respectively.

Example 1. To show the validity of the proposed gain adjusting algorithm of Section 2, it is assumed that $E_o(\theta)$ consists of one harmonic frequency and two non-harmonic frequency components locating at 0° (harmonic frequency), 180° (non-harmonic frequency) and 330° (non-harmonic frequency) (119, 149 and 174 Hz respectively), and the magnitude of each is given as 10, 0.73 and 1 respectively. Note that the three frequency components chosen here are sufficient to set up the practical situation of controlling a hard-disk drive system, since dominant non-repetitive errors often occur at a few non-harmonic frequencies primarily owing to the mechanical characteristics of ball bearings in the spindle motor (Bouchard *et al.*, 1987). $\Delta K_r(1)$ in Algorithm I is chosen as 0.2. Figure 3(a) shows $|E|_m(i)$ and $K_r(i)$ versus iteration number i for cases (i), (ii) and (iii).

It can be seen that $K_r(i)$ converges to 0.41, 0.33 and 0.27 for cases (i), (ii) and (iii) respectively, and $|E|_m(i)$ converges to about 0.9 regardless of the magnitude of $\sqrt{2}\,\delta$ after a few iterations. To investigate whether the finally adjusted value of K_r is sufficient for our purpose, magnitude plots of $E(K_r,\theta)$ for $\theta=0°$, 180° and 330° are sketched in Fig. 3(b) by computing the equation $E(K_r,\theta)=G_{re}(K_r,\theta)E_o(\theta)$ for case (i). It can be shown from Fig. 3(b) that $|E|_m$ is minimum at $K_r=0.41$ for case (i), and thus our proposed gain adjusting algorithm is valid. Furthermore, it is also useful for systems with modelling uncertainties.

Example 2. It can be shown that the proposed gain adjusting algorithm can play a critical role in reducing $|E|_m$ for the modified repetitive control system. For this, it is assumed that the equation of the FIR low-pass filter in Fig. 1(b) is given by $q(z)=\frac{1}{4}(2+z^{-1}+z)$. The same $E_o(\theta)$ and $\Delta K_r(1)$ as in Example 1 are employed. Figure 4(a) shows $|E|_m(i)$ and $K_r(i)$ versus iteration number i for cases (i), (ii) and (iii), and Fig. 4(b) shows magnitude plots of $E(K_r,\theta)$ for $\theta=0°$, 180° and 330° for case (i). It can be seen that K_r converges to 0.52, 0.43 and 0.37 for cases (i), (ii) and (iii) respectively, and $|E|_m$ is minimum at $K_r=0.52$ for case (i). Therefore it is believed that Algorithm 1 also works successfully for modified repetitive control systems, regardless of modelling uncertainties. Here it can be deduced from Fig. 4(b) that at the harmonic frequency, $|E|$ is no longer zero, but is a function of K_r. Furthermore, $|E(K_r)|$ monototically increases as K_r monototically decreases. Thus the simple gain reduction technique proposed by Inoue (1990) cannot be applied.

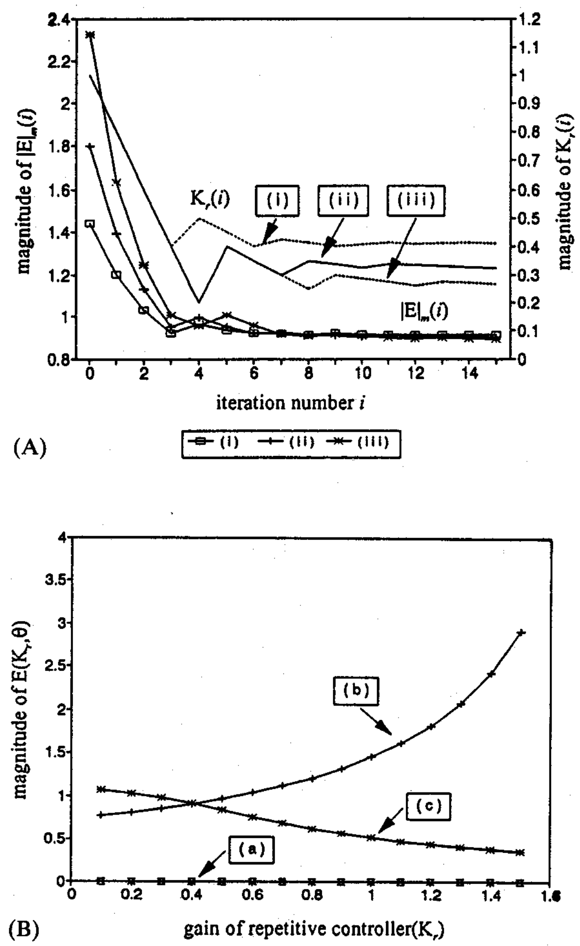


Fig. 3. Example 1, (A) Magnitude plots of $K_r(i)$ and $|E|_m(i)$ versus iteration number i for cases (i), (ii) and (iii). (B) Magnitude plots of $E(K_r, \theta)$ for $\theta = 0°$ (a), 180° (b) and 330° (c).

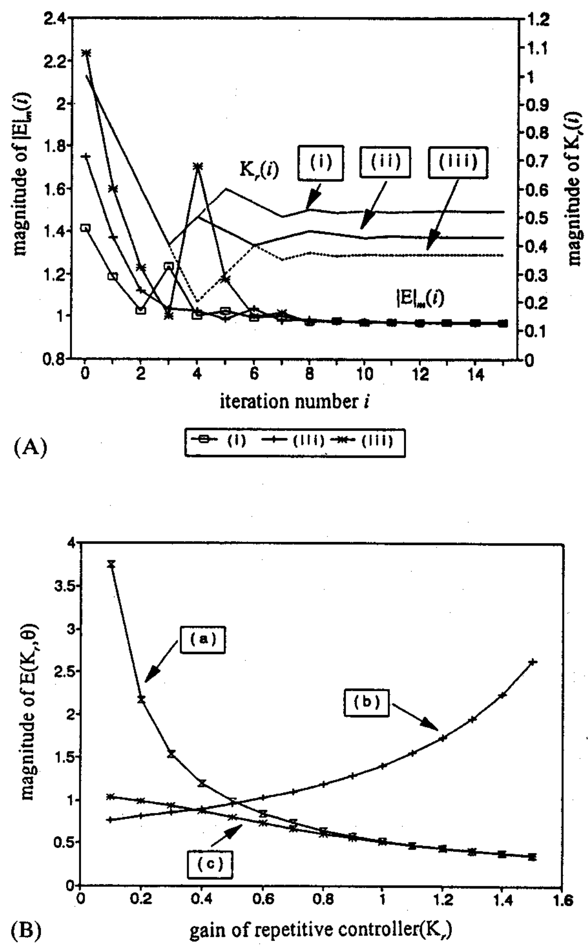


Fig. 4. Example 2. (A) Magnitude plots of $K_r(i)$ and $|E|_m(i)$ versus iteration number i for cases (i), (ii) and (iii). (B) Magnitude plots of $E(K_r, \theta)$ for $\theta = 0°$ (a), 180° (b) and 330° (c).

However, our proposed gain adjusting algorithm guarantees a reduction in $|E|_m$.

3. *Design of a repetitive control system with a higher-order repetitive function*

As an alternative to the gain adjusting algorithm proposed in Section 2, a higher-order repetitive function could be employed to reduce the non-harmonic error components shown in Fig. 1(c). Specifically, consider the jth-order repetitive function $W_j(z)$ given by

$$W_j(z) = \sum_{i=1}^{j} w_i z^{-(i-1)N}, \tag{15}$$

where $\sum_{i=1}^{j} w_i = 1$, $0 \le w_i \le 1$, $|W_j(z)| \le 1$. For the sake of simplicity in observing non-harmonic error behaviour, assume that perfect modelling of $G_{so}(z)$ is available. Then, by a similar approach to that used in deriving $G_{rem}(z)$ in (11), the relative error transfer function $G_{reh}(z)$ can be obtained as

$$G_{reh}(z) = \frac{1 - z^{-N}W_j(z)}{1 - z^{-N}W_j(z) + K_r z^{-N}W_j(z)}. \tag{16}$$

Note that Inoue (1990) proposed the use of equal weightings given as $w_i = 1/j$, for $i = 1, 2, \ldots, j$ for the higher-order repetitive function, which was shown to be the solution minimizing the averaged square of $|G_{reh}(z)|$ over all frequencies. However, it seems to us that the averaged square of $|G_{reh}(z)|$ may not be a good measure, since a particular unwanted noise signal may be overamplified. Thus, instead of the averaged square of $|G_{reh}(z)|$, we shall employ the infinite norm of $|G_{reh}(z)|$ as the performance measure. For this, let $z = e^{j\omega T}$, $\theta = \omega NT$ and $\tilde{W}_j(\theta) = W_j(e^{j\theta/N})$. Then (16) can be written as

$$G_{reh}(K_r, \theta, j, w_i) = \frac{1 - e^{-j\theta}\tilde{W}_j(\theta)}{1 - e^{-j\theta}\tilde{W}_j(\theta) + K_r e^{-j\theta}\tilde{W}_j(\theta)}. \tag{17}$$

For a fixed K_r and a fixed j, define $C_o(j, K_r)$ as

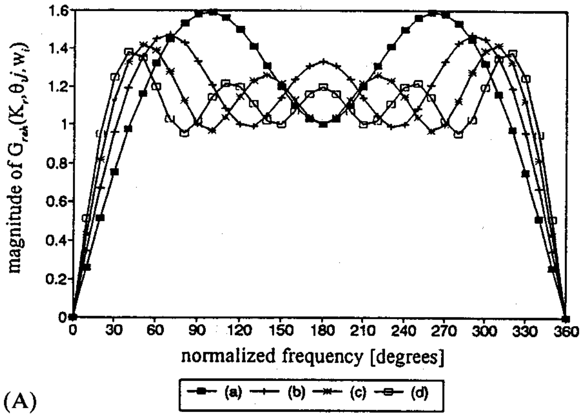
$$C_o(j, K_r) \equiv (w_{o1}, w_{o2}, \ldots, w_{oj}) \equiv \arg\left\{ \min_{0 \le w_1, \ldots, w_j \le 1} \left[\max_{0 \le \theta \le 2\pi} |G_{reh}(K_r, \theta, j, w_i)| \right] \right\}. \tag{18}$$

Here $C_o(j, K_r)$ represents a set of weighting parameters of $\tilde{W}_j(\theta)$ to minimize the infinite norm of the relative error transfer function in (17). Unfortunately, it seems to be difficult to find $C_o(j, K_r)$ analytically therefore we employ a numerical optimization technique called an '*evolution strategy*' (Kasper, 1992), which can minimize and/or maximize an objective function by a stochastic method under some given boundary conditions and constraints. In our case of finding $C_o(j, K_r)$, one equality and one inequality condition are given by

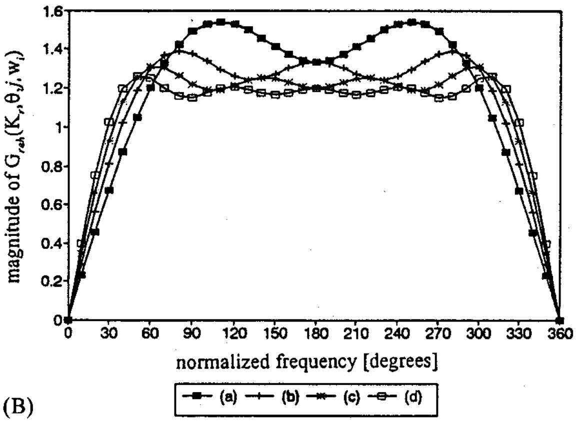
$$\sum_{i=1}^{j} w_{oi} = 1, \quad 0 \le w_{oi} \le 1 \quad (i = 1, 2, \ldots, j).$$

By the evolution strategy method (Kasper, 1992), for $K_r = 1$ and various fixed j, $C_o(j, K_r)$ can be obtained as

$$\begin{aligned} C_o(2,1) &\approx (\tfrac{2}{3}, \tfrac{1}{3}), \\ C_o(3,1) &\approx (\tfrac{3}{6}, \tfrac{2}{6}, \tfrac{1}{6}), \\ C_o(4,1) &\approx (\tfrac{4}{10}, \tfrac{3}{10}, \tfrac{2}{10}, \tfrac{1}{10}), \\ C_o(5,1) &\approx (\tfrac{5}{15}, \tfrac{4}{15}, \tfrac{3}{15}, \tfrac{2}{15}, \tfrac{1}{15}). \end{aligned} \tag{19}$$



(A)



(B)

Fig. 5. (A) Magnitude plots of $G_{\text{reh}}(K_r, \theta, j, w_i)$ for equal weightings for $j = 2$ (a), 3 (b), 4 (c) and 5 (d). (B) Magnitude plots of $G_{\text{reh}}(K_r, \theta, j, w_i)$ for the proposed weightings for $j = 2$ (a), 3 (b), 4 (c) and 5 (d).

To compare the performances for the system with equal weightings with those for the system with our proposed weightings in (19), magnitude plots of $G_{\text{reh}}(K_r, \theta, j, w_i)$ for the cases of equal weightings and proposed weightings in (19) are sketched in Fig. 5, where $K_r = 1$. It can be seen that $G_{\text{reh}}(K_r, \theta, j, w_i)|_m$ can be more effectively reduced by using the proposed weightings than by using equal weightings under the same K_r and j. It is also interesting to note how the infinite norm of the relative error transfer function could be changed by increasing the order of a repetitive function. For this, the infinite norm of $G_{\text{reh}}(K_r, \theta, j, w_i)$ with respect to j for the cases of equal weightings and the proposed weightings are plotted in Fig. 6, where $K_r = 1$. It can be seen that for our proposed weightings, the order of $W_j(z)$ can be chosen as 5, but for equal weightings, the order of $W_j(z)$ should be chosen to be greater than 10 for similar performances to our proposed weightings. Thus it can be concluded that the use of our proposed weightings guarantees a more desirable shape of the relative error transfer function than the use of equal weightings.

4. *Concluding remarks*

Two types of linear digital repetitive control systems have been analysed and designed to reduce the error spectrum, including not only harmonic but also non-harmonic components. First, a novel gain adjusting algorithm was suggested employing the relative error transfer function, in which the gain of the repetitive controller was adjusted to reduce the infinite norm of the error in the frequency domain. It was shown that the proposed gain adjusting algorithm guarantees reduction of the infinite norm of the error in the frequency domain, including both the harmonic

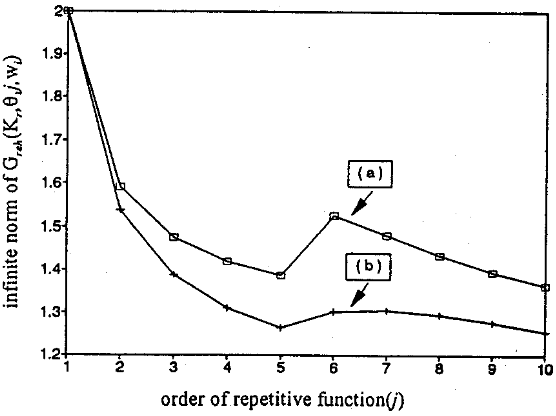


Fig. 6. Infinite norm of $G_{\text{reh}}(K_r, \theta, j, w_i)$ versus j for (a) equal weightings and (b) the proposed weightings.

and non-harmonic frequency components. Significantly, the proposed gain adjusting algorithm was shown to play a critical role in reducing the infinite norm of the error for modified repetitive control systems rather than for conventional repetititive control systems. Secondly, it was shown that when designing a repetitive control system with a higher-order repetitive function as an alternative to a repetitive control system with the proposed gain adjusting algorithm, the use of our proposed unequal weightings of the higher-order repetitive function minimizing the infinite norm of the relative error transfer function was more effective than the use of the equal weightings proposed by Inoue (1990).

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