Performance and $\mathcal{H}_\infty$ Optimality of PID Trajectory Tracking Controller for Lagrangian Systems

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***Abstract*—This paper suggests an inverse optimal proportional–integral–derivative (PID) control design method to track trajectories in Lagrangian systems. The inverse optimal PID controller exists if and only if the Lagrangian system is extended disturbance input-to-state stable. First, we find the Lyapunov function and the control law that satisfy the extended disturbance input-to-state stability by using the characteristics of the Lagrangian system. The control law has a PID control form and satisfies the Hamilton–Jacobi–Isaacs equation. Hence, the $\mathcal{H}_\infty$ inverse optimality of the closed-loop system dynamics is acquired through the PID controller if the conditions for the control law are satisfied. Also, simple coarse/fine performance tuning laws are suggested based on a performance limitation analysis of the inverse optimal PID controller. Selection conditions for gains are proposed as functions of the tuning variable. Experimental results for a typical Lagrangian system show that our analysis provides performance and $\mathcal{H}_\infty$ optimality.**

***Index Terms*—$\mathcal{H}_\infty$ optimality, ISS, performance, PID, tuning rule.**

I. Introduction

THE CONVENTIONAL PID controller for automated machines is widely accepted by industry. According to a survey reported in [1], [2], more than 90% of control loops used in industry use PID. There are many types of PID controllers, e.g., PID plus gravity compensator, PID plus friction compensator, PID plus disturbance observer, etc. The wide acceptance of the PID controller in industry is based on the following advantages: it is easy to use, each term in the PID controller has clear physical meanings (present, past, and predictive), and it can be used irrespective of the system dynamics. A $\mathcal{H}_\infty$ optimal controller that is robust and performs well has been developed for nonlinear mechanical control systems; however, it has not been widely accepted in industry since it is not immediately clear which partial differential equations should be solved. To transfer $\mathcal{H}_\infty$ control theory for industrial applications, it is worthwhile to describe the relationship between $\mathcal{H}_\infty$ control and PID. In this paper, we analyze the $\mathcal{H}_\infty$ optimality and performance of a PID controller, especially for Lagrangian systems.

Most industrial mechanical systems can be described by the Lagrangian equation of motion. Conventional PID trajectory tracking controllers are used because they provide very effective position control in Lagrangian systems. Unfortunately, they lack an asymptotic stability proof. Under some conditions with PID gains, the global (or semi-global) asymptotic stability of a PID set-point regulation controller was proved by [3]–[7] for robotic manipulator systems without external disturbances. However, they did not deal with a PID trajectory tracking controller, but rather a set-point regulation one. Also, the effect of PID gains on the system performance in view of $\mathcal{H}_\infty$ optimality was not considered.

In optimal control theories, nonlinear $\mathcal{H}_\infty$ control methods that are robust and perform well have been proposed over the last decade. The basic control law theories are found in two papers [8], [9]: one describing the full state feedback case, and the other considering the output feedback case. However, the partial differential Hamilton–Jacobi–Isaacs (HJI) equation must still be solved in a nonlinear $\mathcal{H}_\infty$ controller. This is not a trivial problem. There have been several attempts to solve the HJI equation. The approximation method was used in [10] to obtain an approximate solution to the HJI equation for Lagrangian systems. The concept of extended disturbances, including system error dynamics, was developed in [11]–[14] to solve the HJI equation. Finally, the Lyapunov equation was solved instead of the HJI equation by Su *et al.* [15], who suggested that a nonlinear $\mathcal{H}_\infty$ controller can only be attained in this manner.

The following notations are used in this paper. The $\mathcal{L}_2$ norm is defined by $\|\boldsymbol{x}(t)\| = \sqrt{\int_0^t \boldsymbol{x}(\tau)^T \boldsymbol{x}(\tau)\, d\tau}$ and the Euclidian norm is defined by $|\boldsymbol{x}(t)| = \sqrt{\boldsymbol{x}(t)^T \boldsymbol{x}(t)}$. A continuous function γ: $[0, a) \rightarrow \Re_+$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$. The function is said to belong to class $\mathcal{K}_\infty$ if $a = \infty$ and $\gamma(r) \rightarrow \infty$ as $r \rightarrow \infty$. Also, a continuous function β: $[0, a) \times \Re_+ \rightarrow \Re_+$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r, and for each fixed r the mapping $\beta(r, s) \rightarrow 0$ as $s \rightarrow \infty$.

The paper is organized as follows. The next section deals with the state-space representation used for trajectory tracking in a Lagrangian system. In Section III, the $\mathcal{H}_\infty$ optimality of a PID controller is proved by inversely finding the $\mathcal{H}_\infty$ performance index from the PID control law. The coarse/fine performance tuning laws and gain selection methods are proposed in Section IV. The experimental results and concluding remarks are given in Sections V and VI, respectively.

II. State-Space Description of Lagrangian Systems

In general, the mechanical systems can be described by the Lagrangian equation of motion. If the mechanical system with

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n degrees of freedom is represented by n generalized configuration coordinates $\boldsymbol{q} = [q_1, q_2, \ldots, q_n]^T \in \Re^n$, then the Lagrangian system is described as

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{d}(t) = \boldsymbol{\tau} \qquad (1)$$

where $\boldsymbol{M}(\boldsymbol{q}) \in \Re^{n\times n}$ is Inertia matrix, $\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} \in \Re^n$ is the Coriolis and centrifugal torque vector, $\boldsymbol{g}(\boldsymbol{q}) \in \Re^n$ represents the gravitational torque vector, $\boldsymbol{\tau} \in \Re^n$ is the control input torque vector, and $\boldsymbol{d}(t)$ represent the unknown external disturbances. Disturbances exerted on the system can be caused by the friction nonlinearity, parameter perturbation, etc. Also, the extended disturbance can be defined for the trajectory tracking control, including the external disturbance, as follows:

$$\begin{aligned}\boldsymbol{w}\left(t, \dot{\boldsymbol{e}}, \boldsymbol{e}, \int \boldsymbol{e}\right) &= \boldsymbol{M}(\boldsymbol{q})(\ddot{\boldsymbol{q}}_d + \boldsymbol{K}_P\dot{\boldsymbol{e}} + \boldsymbol{K}_I\boldsymbol{e}) \\ &+ \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\left(\dot{\boldsymbol{q}}_d + \boldsymbol{K}_P\boldsymbol{e} + \boldsymbol{K}_I\int \boldsymbol{e}\right) + \boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{d}(t) \qquad (2)\end{aligned}$$

where $\boldsymbol{K}_P, \boldsymbol{K}_I$ are the diagonal constant matrices, $\boldsymbol{e} = \boldsymbol{q}_d - \boldsymbol{q}$ is the configuration error and the desired configurations $(\boldsymbol{q}_d, \dot{\boldsymbol{q}}_d, \ddot{\boldsymbol{q}}_d)$ are functions of time. Hence, the extended disturbance $\boldsymbol{w}$ is the function of time, configuration error, its derivative and integral because $\boldsymbol{q}(=\boldsymbol{q}_d - \boldsymbol{e})$ and $\dot{\boldsymbol{q}}(= \dot{\boldsymbol{q}}_d - \dot{\boldsymbol{e}})$ are the function of time, configuration error, and its derivative. If the extended disturbance defined above is used in the Lagrangian system of (1), then the trajectory tracking system model can be rewritten as

$$\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{s}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\boldsymbol{s} = \boldsymbol{w}\left(t, \dot{\boldsymbol{e}}, \boldsymbol{e}, \int \boldsymbol{e}\right) + \boldsymbol{u} \qquad (3)$$

where $\boldsymbol{u} = -\boldsymbol{\tau}$ and $\boldsymbol{s} = \dot{\boldsymbol{e}} + \boldsymbol{K}_P\boldsymbol{e} + \boldsymbol{K}_I \int \boldsymbol{e}\, dt$.

If the state vector is defined as $\boldsymbol{x} = [\int \boldsymbol{e}^T, \boldsymbol{e}^T, \dot{\boldsymbol{e}}^T]^T \in \Re^{3n}$ for the tracking system model (3), then the state-space representation of Lagrangian system can be obtained as follows:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}(\boldsymbol{x}, t)\boldsymbol{x} + \boldsymbol{B}(\boldsymbol{x}, t)\boldsymbol{w} + \boldsymbol{B}(\boldsymbol{x}, t)\boldsymbol{u} \qquad (4)$$

where

$$\boldsymbol{A}(\boldsymbol{x}, t) = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{M}^{-1}\boldsymbol{C}\boldsymbol{K}_I & -\boldsymbol{M}^{-1}\boldsymbol{C}\boldsymbol{K}_P - \boldsymbol{K}_I & -\boldsymbol{M}^{-1}\boldsymbol{C} - \boldsymbol{K}_P \end{bmatrix}$$

and

$$\boldsymbol{B}(\boldsymbol{x}, t) = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{M}^{-1} \end{bmatrix}.$$

This is one of the generic forms defined in [14] for a Lagrangian system. A characteristic of the Lagrangian system is that the equality $(\dot{\boldsymbol{M}} = \boldsymbol{C} + \boldsymbol{C}^T)$ is always satisfied. This characteristic offers the clue to solving the inverse optimal problem for above the Lagrangian system.

Remark 1: If the controller stabilizes the trajectory tracking system model (4), then it makes the original system (1) stable because the boundedness of a state vector $\boldsymbol{x}$ implies those of $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$. However, the converse is not true.

Remark 2: For the set-point regulation control, the system model (1) can be rewritten by using the state vector $\dot{\boldsymbol{q}}$ as follows:

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \boldsymbol{w}_1(t, \boldsymbol{q}) + \boldsymbol{\tau} \qquad (5)$$

where $\boldsymbol{w}_1(t, \boldsymbol{q}) = -\boldsymbol{g}(\boldsymbol{q}) - \boldsymbol{d}(t)$. On the other hand, for the trajectory tracking control, we obtained the system model (3) by using the state vector $\boldsymbol{s}$. Here, the above two system models (3) and (5) show the same dynamic characteristics such as $\boldsymbol{M}(\boldsymbol{q})^{-1}\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})$.

III. ISS and $\mathcal{H}_\infty$ Optimality of PID Control

Among the stability theories for control systems, the notion of input-to-state stability (ISS) in [16]–[20] is more convenient to deal with the disturbance input than other theories. When there exist unknown bounded inputs such as perturbations and external disturbances acting on systems, the behavior of the system should remain bounded. Also, when the set of inputs including the control, perturbation, and disturbance go to zero, the behavior of the system tends toward the equilibrium point. This ISS notion is helpful to understand the effect of inputs on system states. Moreover, Krstic *et al.* showed in [21] and [22] that the backstepping controller designed using the ISS notion is optimal for the performance index found inversely from the controller. This has offered a useful insight from which we can show the $\mathcal{H}_\infty$ optimality of PID control for Lagrangian systems. The basic characteristics and properties on the ISS are summarized below.

The control system is said to be extended disturbance ISS if there exist a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ such that the solution for (4) exists for all $t \geq 0$ and satisfies

$$|\boldsymbol{x}(t)| \leq \beta(|\boldsymbol{x}(0)|, t) + \gamma\left(\sup_{0\leq\tau\leq t} |\boldsymbol{w}(\tau)|\right)$$

for an initial state vector $\boldsymbol{x}(0)$ and for an extended disturbance vector $\boldsymbol{w}(\cdot)$ piecewise continuous and bounded on $[0, \infty)$. In particular, the ISS becomes available by using the Lyapunov function. For the system (4), there exist a smooth positive definite radially unbounded function $V(\boldsymbol{x}, t)$, a class $\mathcal{K}_\infty$ function γ_1, and a class $\mathcal{K}$ function γ_2 such that the following dissipativity inequality is satisfied:

$$\dot{V} \leq -\gamma_1(|\boldsymbol{x}|) + \gamma_2(|\boldsymbol{w}|) \qquad (6)$$

if and only if the system is ISS, where $\dot{V}$ represents the total derivative for the Lyapunov function. Also, suppose that there exists a function $V(\boldsymbol{x}, t)$ such that the following implication holds for all $\boldsymbol{x}$ and $\boldsymbol{w}$:

$$|\boldsymbol{x}| \geq \rho(|\boldsymbol{w}|) \Rightarrow \dot{V} \leq -\gamma_3(|\boldsymbol{x}|) \qquad (7)$$

where ρ and γ_3 are class $\mathcal{K}_\infty$ functions. Then, the system is ISS and even we can say the globally asymptotic stability (GAS) if the unknown disturbance input satisfies the condition $|\boldsymbol{x}| \geq \rho(|\boldsymbol{w}|)$ for a state vector. However, we do not know whether the extended disturbance $\boldsymbol{w}$ satisfies the condition or not, hence, only ISS is proved. The above properties on ISS will be utilized in the following sections.

A. ISS-CLF for Lagrangian Systems

To show the ISS for Lagrangian system, we should find the Lyapunov function and control law. However, there can be many control laws satisfying the ISS and the disturbance is basically unknown. Hence, we need the definition which can bring the Lyapunov function and the unique control input under the assumption on the unknown disturbance. Using the ISS, the input-to-state stabilizable control Lyapunov function, in short ISS-CLF, was defined by [21]. The regular definition for ISS-CLF is as follows: a smooth positive definite radially unbounded function $V(\boldsymbol{x}, t)$: $\Re^{3n} \times \Re_+ \rightarrow \Re_+$ is called an ISS-CLF for (4) if there exists a class $\mathcal{K}_\infty$ function ρ such that the following implication holds for all $\boldsymbol{x} \neq \mathbf{0}$ and all $\boldsymbol{w}$:

$$|\boldsymbol{x}| \geq \rho(|\boldsymbol{w}|) \quad \Rightarrow \quad \inf_{\boldsymbol{u}} \dot{V} < 0. \tag{8}$$

The following Theorem suggests both the control law derived from ISS and an ISS-CLF for a Lagrangian system. Here, we show that the modified form of the Lyapunov function suggested by [14] is an ISS-CLF under two conditions.

Theorem 1: Let $\boldsymbol{s} \triangleq \dot{\boldsymbol{e}} + \boldsymbol{K}_P\boldsymbol{e} + \boldsymbol{K}_I \int \boldsymbol{e}\, dt \in \Re^n$. If the Lagrangian system (4) is extended disturbance ISS, then the control law should have the following form with $\alpha \geq 1/2$:

$$\boldsymbol{u} = -\alpha\boldsymbol{K}\boldsymbol{s} - \rho^{-1}(|\boldsymbol{x}|)\frac{\boldsymbol{s}}{|\boldsymbol{s}|} \tag{9}$$

and $V(\boldsymbol{x}, t) = (1/2)\boldsymbol{x}^T\boldsymbol{P}(\boldsymbol{x}, t)\boldsymbol{x}$ is an ISS-CLF with $\alpha = 1/2$, where

$$\boldsymbol{P}(\boldsymbol{x}, t) = \begin{bmatrix} \boldsymbol{K}_I\boldsymbol{M}\boldsymbol{K}_I + \boldsymbol{K}_I\boldsymbol{K}_P\boldsymbol{K} & \boldsymbol{K}_I\boldsymbol{M}\boldsymbol{K}_P + \boldsymbol{K}_I\boldsymbol{K} & \boldsymbol{K}_I\boldsymbol{M} \\ \boldsymbol{K}_P\boldsymbol{M}\boldsymbol{K}_I + \boldsymbol{K}_I\boldsymbol{K} & \boldsymbol{K}_P\boldsymbol{M}\boldsymbol{K}_P + \boldsymbol{K}_P\boldsymbol{K} & \boldsymbol{K}_P\boldsymbol{M} \\ \boldsymbol{M}\boldsymbol{K}_I & \boldsymbol{M}\boldsymbol{K}_P & \boldsymbol{M} \end{bmatrix} \tag{10}$$

under the following two conditions for $\boldsymbol{P}$:

1) $\boldsymbol{K}, \boldsymbol{K}_P, \boldsymbol{K}_I > \mathbf{0}$ constant diagonal matrices;
2) $\boldsymbol{K}_P^2 > 2\boldsymbol{K}_I$.

Proof: First, we show that the Lyapunov matrix $\boldsymbol{P}$ (10) is positive definite. If we manipulate the Lyapunov function as follows:

$$\begin{aligned} V(\boldsymbol{x}, t) &= \tfrac{1}{2}\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x} \\ &= \tfrac{1}{2}\boldsymbol{s}^T\boldsymbol{M}\boldsymbol{s} + \tfrac{1}{2}\begin{bmatrix} \int \boldsymbol{e} \\ \boldsymbol{e} \end{bmatrix}^T \begin{bmatrix} \boldsymbol{K}_I\boldsymbol{K}_P\boldsymbol{K} & \boldsymbol{K}_I\boldsymbol{K} \\ \boldsymbol{K}_I\boldsymbol{K} & \boldsymbol{K}_P\boldsymbol{K} \end{bmatrix} \begin{bmatrix} \int \boldsymbol{e} \\ \boldsymbol{e} \end{bmatrix} \end{aligned}$$

then we can see that the Lyapunov function is positive definite under conditions 1 and 2 except $\boldsymbol{x} = \mathbf{0}$ because the Inertia matrix $\boldsymbol{M}$ is positive definite. Second, the total derivative of Lyapunov function is given by

$$\dot{V} = V_t + \boldsymbol{V}_x\boldsymbol{A}\boldsymbol{x} + \boldsymbol{V}_x\boldsymbol{B}\boldsymbol{w} + \boldsymbol{V}_x\boldsymbol{B}\boldsymbol{u}$$

and its components can be calculated using $\dot{\boldsymbol{M}} - \boldsymbol{C}^T - \boldsymbol{C} = \mathbf{0}$ as follows:

$$\begin{aligned} V_t + \boldsymbol{V}_x\boldsymbol{A}\boldsymbol{x} &= \tfrac{1}{2}\boldsymbol{x}^T\left(\dot{\boldsymbol{P}} + \boldsymbol{P}\boldsymbol{A} + \boldsymbol{A}^T\boldsymbol{P}\right)\boldsymbol{x} \\ &= \tfrac{1}{2}\boldsymbol{x}^T\begin{bmatrix} \mathbf{0} & \boldsymbol{K}_I\boldsymbol{K}_P\boldsymbol{K} & \boldsymbol{K}_I\boldsymbol{K} \\ \boldsymbol{K}_I\boldsymbol{K}_P\boldsymbol{K} & 2\boldsymbol{K}_I\boldsymbol{K} & \boldsymbol{K}_P\boldsymbol{K} \\ \boldsymbol{K}_I\boldsymbol{K} & \boldsymbol{K}_P\boldsymbol{K} & \mathbf{0} \end{bmatrix}\boldsymbol{x} \\ &= \tfrac{1}{2}\left(\boldsymbol{s}^T\boldsymbol{K}\boldsymbol{s} - \int \boldsymbol{e}^T\boldsymbol{K}_I^2\boldsymbol{K}\int \boldsymbol{e}\right. \\ &\qquad \left. -\boldsymbol{e}^T(\boldsymbol{K}_P^2 - 2\boldsymbol{K}_I)\boldsymbol{K}\boldsymbol{e} - \dot{\boldsymbol{e}}^T\boldsymbol{K}\dot{\boldsymbol{e}}\right), \end{aligned} \tag{11}$$

and

$$\boldsymbol{V}_x\boldsymbol{B} = \boldsymbol{x}^T\boldsymbol{P}\boldsymbol{B} = \boldsymbol{x}^T[\boldsymbol{K}_I, \boldsymbol{K}_P, \boldsymbol{I}]^T = \boldsymbol{s}^T. \tag{12}$$

Now, the total derivative of the Lyapunov function is calculated by using (11) and (12) as follows:

$$\begin{aligned} \dot{V} &= \tfrac{1}{2}\left(\boldsymbol{s}^T\boldsymbol{K}\boldsymbol{s} - \int \boldsymbol{e}^T\boldsymbol{K}_I^2\boldsymbol{K}\int \boldsymbol{e}\right. \\ &\qquad \left. -\boldsymbol{e}^T(\boldsymbol{K}_P^2 - 2\boldsymbol{K}_I)\boldsymbol{K}\boldsymbol{e} - \dot{\boldsymbol{e}}^T\boldsymbol{K}\dot{\boldsymbol{e}}\right) + \boldsymbol{s}^T\boldsymbol{w} + \boldsymbol{s}^T\boldsymbol{u} < 0 \\ &\leftrightarrow \tfrac{1}{2}\boldsymbol{s}^T\boldsymbol{K}\boldsymbol{s} + \boldsymbol{s}^T\boldsymbol{w} + \boldsymbol{s}^T\boldsymbol{u} \\ &< \tfrac{1}{2}\left(\int \boldsymbol{e}^T(\boldsymbol{K}_I^2\boldsymbol{K})\int \boldsymbol{e} + \boldsymbol{e}^T(\boldsymbol{K}_P^2 - 2\boldsymbol{K}_I)\boldsymbol{K}\boldsymbol{e} + \dot{\boldsymbol{e}}^T\boldsymbol{K}\dot{\boldsymbol{e}}\right). \end{aligned} \tag{13}$$

Let us consider only the right-hand side of (13). Then we can see that it is always positive definite except $\boldsymbol{x} = \mathbf{0}$ under conditions 1 and 2. Also, if the condition $|\boldsymbol{x}| \geq \rho(|\boldsymbol{w}|)$ of (7) is utilized, then the left-hand side of (13) has the following form:

$$\begin{aligned} \tfrac{1}{2}\boldsymbol{s}^T\boldsymbol{K}\boldsymbol{s} + \boldsymbol{s}^T\boldsymbol{w} + \boldsymbol{s}^T\boldsymbol{u} &\leq \tfrac{1}{2}\boldsymbol{s}^T\boldsymbol{K}\boldsymbol{s} + |\boldsymbol{s}||\boldsymbol{w}| + \boldsymbol{s}^T\boldsymbol{u} \\ &\leq \tfrac{1}{2}\boldsymbol{s}^T\boldsymbol{K}\boldsymbol{s} + |\boldsymbol{s}|\rho^{-1}(|\boldsymbol{x}|) + \boldsymbol{s}^T\boldsymbol{u}. \end{aligned}$$

Here, the above equation should at least be negative semi-definite to satisfy the ISS of (7). Hence, we obtain the control law (9) with the condition $\alpha \geq 1/2$. Also, since the infimum among the control inputs that satisfy (13) is achieved at $\alpha = 1/2$, we can know that the definition (8) is always satisfied with $\alpha = 1/2$ for all $\boldsymbol{x} \neq \mathbf{0}$. Third, the $V(\boldsymbol{x}, t)$ is a differentiable and radially unbounded function because $V(\boldsymbol{x}, t) \rightarrow \infty$ as $\boldsymbol{x} \rightarrow \infty$. Therefore, we conclude that $V(\boldsymbol{x}, t)$ is an ISS-CLF with $\alpha = 1/2$ for the Lagrangian systems. ■

An important characteristics of controller (9) is that it has the PID control type as follows:

$$\boldsymbol{u} = -\left(\alpha\boldsymbol{K} + \frac{\rho^{-1}(|\boldsymbol{x}|)}{|\boldsymbol{s}|}\boldsymbol{I}\right)\left(\dot{\boldsymbol{e}} + \boldsymbol{K}_P\boldsymbol{e} + \boldsymbol{K}_I\int \boldsymbol{e}\, dt\right). \tag{14}$$

Another characteristic of the above controller is that it can be rewritten as the optimal control type of $\boldsymbol{u} = -\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{x}$ by letting

$$\boldsymbol{R}(\boldsymbol{x}) = \left(\alpha\boldsymbol{K} + \frac{\rho^{-1}(|\boldsymbol{x}|)}{|\boldsymbol{s}|}\boldsymbol{I}\right)^{-1}$$

because $\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{x} = \dot{\boldsymbol{e}} + \boldsymbol{K}_P\boldsymbol{e} + \boldsymbol{K}_I \int \boldsymbol{e}\,dt$ as shown in (12). Strictly speaking, the controller (14) is not a conventional PID one since it includes the unknown function $\rho^{-1}(|\boldsymbol{x}|)$. In the following section, we will find the bounds and meaning of $\rho^{-1}(|\boldsymbol{x}|)$ in a viewpoint of the optimality of $\mathcal{H}_\infty$ control.

B. $\mathcal{H}_\infty$ Optimality of PID Control Law

To state the optimality of the nonlinear $\mathcal{H}_\infty$ control system suggested in [8] and [9], the HJI equation which is derived from the direct optimization for the performance index should be solved, but the solution of HJI equation is too hard to obtain for the general systems, including Lagrangian system, because it is the multi-variable partial differential equation. To overcome this difficulty of a direct optimization, Krstic *et al.* showed in [21] that the inverse optimal problem is solvable if the system is disturbance input-to-state stable. Also, Park *et al.* showed in [14] that the nonlinear $\mathcal{H}_\infty$ control problem for robotic manipulators can be solved using the characteristics of the Lagrangian system. The HJI equation for the Lagrangian system and its analytic solution were suggested by [12], but it dealt with the modified computed torque controller form, not a PID controller type.

Now, we will show the $\mathcal{H}_\infty$ optimality of the PID control type for Lagrangian systems by using a control law in Theorem 1. Consider the general $\mathcal{H}_\infty$ performance index (PI) in the following form:

$$PI(t,\,\boldsymbol{x},\,\boldsymbol{u},\,\boldsymbol{w}) = \lim_{t\to\infty}\left[2V(\boldsymbol{x}(t),\,t) + \int_0^t (\boldsymbol{x}^T\boldsymbol{Q}(\boldsymbol{x})\boldsymbol{x} + \boldsymbol{u}^T\boldsymbol{R}(\boldsymbol{x})\boldsymbol{u} - \gamma^2\boldsymbol{w}^T\boldsymbol{w})\,d\tau\right] \tag{15}$$

where $\boldsymbol{Q}(\boldsymbol{x})$ is a state weighting matrix, $\boldsymbol{R}(\boldsymbol{x})$ is the control input weighting, and γ means $\mathcal{L}_2$-gain. Also, the HJI equation is derived from the optimization for the $\mathcal{H}_\infty$ performance index, with the Lagrangian system constraint of (4), as follows:

$$\text{HJI} = \dot{\boldsymbol{P}} + \boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} - \boldsymbol{P}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P} + \frac{1}{\gamma^2}\boldsymbol{P}\boldsymbol{B}\boldsymbol{B}^T\boldsymbol{P} + \boldsymbol{Q} = \boldsymbol{0}. \tag{16}$$

As a matter of fact, the above HJI equation is equal to the differential Riccati equation for a linear multivariable time-varying system. The HJI equation (16) plays important roles which give the $\mathcal{H}_\infty$ optimality and stability to the control system. In the next Theorem, we show that the PID control law can be the minimum solution of $\mathcal{H}_\infty$ performance index, and it is inverse optimal in that the state-weighting matrix $\boldsymbol{Q}(\boldsymbol{x})$ and control input one $\boldsymbol{R}(\boldsymbol{x})$ can be found from the gains of controller and even the HJI equation can be obtained from $\boldsymbol{Q}(\boldsymbol{x})$.

Theorem 2: For a given Lagrangian system (4), suppose that there exists an ISS-CLF in Theorem 1. If the PID control law (14) with the following form:

$$\boldsymbol{u} = -\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{x} \tag{17}$$

is utilized with the conditions

1) $\alpha = 1$;
2) $\rho^{-1}(|\boldsymbol{x}|) \geq (1/\gamma^2)|\boldsymbol{s}|$;

then the controller (17) is a solution of the minimization problem for $\mathcal{H}_\infty$ performance index (15) using

$$\boldsymbol{Q}(\boldsymbol{x}) = -\left(\dot{\boldsymbol{P}} + \boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} - \boldsymbol{P}\boldsymbol{B}\boldsymbol{K}\boldsymbol{B}^T\boldsymbol{P}\right) \tag{18}$$

$$\boldsymbol{R}(\boldsymbol{x}) = \left(\boldsymbol{K} + \frac{\rho^{-1}(|\boldsymbol{x}|)}{|\boldsymbol{s}|}\boldsymbol{I}\right)^{-1}. \tag{19}$$

Proof: First, we show that the matrix $\boldsymbol{Q}(\boldsymbol{x})$ of (18) is positive definite and constant matrix. Let us obtain the state-weighting matrix $\boldsymbol{Q}(\boldsymbol{x})$ using (11) and (12) in proof of Theorem 1, then $\boldsymbol{Q}(\boldsymbol{x})$ is acquired as the following constant matrix:

$$\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{K}_I^2\boldsymbol{K} & 0 & 0 \\ 0 & (\boldsymbol{K}_P^2 - 2\boldsymbol{K}_I)\boldsymbol{K} & 0 \\ 0 & 0 & \boldsymbol{K} \end{bmatrix}.$$

Hence, $\boldsymbol{Q}$ is a positive definite and constant matrix. This was proved by [13] for the first time. Second, we prove the $\mathcal{H}_\infty$ optimality inversely by showing that a PID control type (17) achieves the minimum of the $\mathcal{H}_\infty$ performance index. The first condition of $\alpha = 1$, not $1/2$, makes it possible to solve the optimization problem, in other words, the optimal α is two times the value obtained by the definition of ISS-CLF in Theorem 1. This fact was proved by [21] for the first time. If we put $\boldsymbol{Q}$ into the performance index and use $\boldsymbol{K} = \boldsymbol{R}^{-1}(\boldsymbol{x}) - (\rho^{-1}(|\boldsymbol{x}|)/|\boldsymbol{s}|)\boldsymbol{I}$ of (19), then we can manipulate the performance index as follows:

$$\begin{aligned}
&PI(t,\,\boldsymbol{x},\,\boldsymbol{u},\,\boldsymbol{w})\\
&= \lim_{t\to\infty}\Bigg[2V(\boldsymbol{x}(t),\,t) - \int_0^t \boldsymbol{x}^T\left[\dot{\boldsymbol{P}} + \boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} - \boldsymbol{P}\boldsymbol{B}\boldsymbol{K}\boldsymbol{B}^T\boldsymbol{P}\right]\boldsymbol{x}\,d\tau + \int_0^t (\boldsymbol{u}^T\boldsymbol{R}(\boldsymbol{x})\boldsymbol{u} - \gamma^2\boldsymbol{w}^T\boldsymbol{w})\,d\tau\Bigg]\\
&= \lim_{t\to\infty}\Bigg[2V(\boldsymbol{x}(t),\,t) - \int_0^t \Big(\boldsymbol{x}^T\left[\dot{\boldsymbol{P}} + \boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}\right]\boldsymbol{x} + 2\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{B}\boldsymbol{u} + 2\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{B}\boldsymbol{w}\Big)\,d\tau\\
&\qquad + \int_0^t (\boldsymbol{u}^T\boldsymbol{R}(\boldsymbol{x})\boldsymbol{u} + 2\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{B}\boldsymbol{u} + \boldsymbol{x}^T\boldsymbol{P}\boldsymbol{B}\boldsymbol{K}\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{x})\,d\tau + \int_0^t (2\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{B}\boldsymbol{w} - \gamma^2\boldsymbol{w}^T\boldsymbol{w})\,d\tau\Bigg]\\
&= \lim_{t\to\infty}\Bigg[2V(\boldsymbol{x}(t),\,t) - 2\int_0^t \dot{V}\,d\tau + \int_0^t (\boldsymbol{u} + \boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{x})^T\boldsymbol{R}(\boldsymbol{u} + \boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{x})\,d\tau\\
&\qquad - \gamma^2\int_0^t \left(\boldsymbol{w} - \frac{1}{\gamma^2}\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{x}\right)^T\left(\boldsymbol{w} - \frac{1}{\gamma^2}\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{x}\right)d\tau - \int_0^t \left(\frac{\rho^{-1}(|\boldsymbol{x}|)}{|\boldsymbol{s}|} - \frac{1}{\gamma^2}\right)\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{B}\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{x}\,d\tau\Bigg]\\
&= 2V(\boldsymbol{x}(0),\,0) + \int_0^\infty (\boldsymbol{u} + \boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{x})^T \cdot \boldsymbol{R}(\boldsymbol{u} + \boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{x})\,d\tau - \gamma^2\int_0^\infty \left|\boldsymbol{w} - \frac{1}{\gamma^2}\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{x}\right|^2 d\tau\\
&\qquad - \int_0^\infty \left(\frac{\rho^{-1}(|\boldsymbol{x}|)}{|\boldsymbol{s}|} - \frac{1}{\gamma^2}\right)|\boldsymbol{s}|^2\,d\tau.
\end{aligned} \tag{20}$$

From (20), we can see that the minimum for the $\mathcal{H}_\infty$ performance index is achieved in the case that the control law is (17). Also, the worst-case disturbance is given by

$$\boldsymbol{w}^* = \frac{1}{\gamma^2}\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{x}$$

and $|\boldsymbol{w}^*| = (1/\gamma^2)|\boldsymbol{s}|$. The second condition of $\rho^{-1}(|\boldsymbol{x}|) \geq (1/\gamma^2)|\boldsymbol{s}|$ should be satisfied for the minimization of (20). Therefore, we conclude that the PID control law (17) minimizes the $\mathcal{H}_\infty$ performance index (15) using the given $\boldsymbol{Q}$ and $\boldsymbol{R}(\boldsymbol{x})$. ■

Remark 3: The condition 2 in Theorem 2 is the design guideline of function $\rho^{-1}(|\boldsymbol{x}|)$ to be the $\mathcal{H}_\infty$ optimal controller. As a matter of fact, it implies that $\rho^{-1}(|\boldsymbol{x}|)$ should not be smaller than at least the magnitude of worst-case disturbance. Here, if we choose the magnitude of worst-case disturbance for the function $\rho^{-1}(|\boldsymbol{x}|)$, in other words, $\rho^{-1}(|\boldsymbol{x}|) = |\boldsymbol{w}^*| = (1/\gamma^2)|\boldsymbol{s}|$, then the PID control law (17) recovers fortunately the static gain PID one because the matrix $\boldsymbol{R}(\boldsymbol{x})$ of (19) becomes a constant matrix as follows:

$$\boldsymbol{R} = \left(\boldsymbol{K} + \frac{1}{\gamma^2}\boldsymbol{I}\right)^{-1}.$$

Also, the matrix $\boldsymbol{Q}$ of (18) implies the HJI equation of (16), though the HJI is not explicitly utilized to show the $\mathcal{H}_\infty$ optimality of a PID control law in Theorem 2.

In a viewpoint of an optimal control theory, the magnitude of a state-weighting $\boldsymbol{Q}$ has a relation with system errors, e.g., if we enlarge the magnitude of a state-weighting matrix by four times, then the control performance will be approximately enhanced by two times, in other words, the error will be approximately reduced by half. This property will be shown through the experimental results later. Therefore, if we are to reduce the error of a system, we should enlarge the magnitude of a matrix $\boldsymbol{K}$ in the state weighting. However, it reduces the magnitude of a control input weighting matrix $\boldsymbol{R}$ and produces the bigger control effort. Conversely, if we reduce the magnitude of $\boldsymbol{K}$, then the smaller control effort is required and the bigger error is generated. The common $\boldsymbol{K}$ in the state weighting and the control input weighting has the tradeoff characteristics between the system performance and control effort. Also, it seems that the $\mathcal{L}_2$-gain γ has no the effect on state weighting and affects only the control input weighting, but it does affect the control performance by increasing the robustness for disturbances.

IV. Performance of Inverse Optimal PID Control

In the previous section, the $\mathcal{H}_\infty$ optimality of PID controller for the performance index was shown through Theorems 1 and 2 and Remark 3. On the other hand, the performance prediction of the PID controller is also an important subject. In this section, we consider the performance of PID controller in viewpoints of its limitation and tuning law. To begin with, we define the inverse optimal PID controller from the static gain one in Remark 3 and summarize its design conditions in the following Theorem.

Theorem 3: If the inverse optimal PID controller

$$\boldsymbol{\tau} = \left(\boldsymbol{K} + \frac{1}{\gamma^2}\boldsymbol{I}\right)\left(\dot{\boldsymbol{e}} + \boldsymbol{K}_P\boldsymbol{e} + \boldsymbol{K}_I\int \boldsymbol{e}\right) \tag{21}$$

satisfying the following conditions:

1) $\boldsymbol{K}, \boldsymbol{K}_P, \boldsymbol{K}_I > \boldsymbol{0}$, constant diagonal matrices;
2) $\boldsymbol{K}_P^2 > 2\boldsymbol{K}_I$;
3) $\gamma > 0$;

is applied to the Lagrangian system (4), then the closed-loop control system is extended disturbance ISS.

Proof: The conditions 1 and 2 assure the existence of the Lyapunov matrix $\boldsymbol{P} > \boldsymbol{0}$ by Theorem 1 and condition 3 means $\mathcal{L}_2$-gain in the $\mathcal{H}_\infty$ performance index. If the inverse optimal PID controller (21) is applied to the Lagrangian system (4), then we know that the HJI equation of (16) is satisfied by Remark 3. Therefore, along the solution trajectory of (4) with the inverse optimal PID control law, we get the time derivative of the Lyapunov function

$$\begin{aligned}\dot{V} &= V_t + \boldsymbol{V}_x\boldsymbol{A}\boldsymbol{x} + \boldsymbol{V}_x\boldsymbol{B}\boldsymbol{u} + \boldsymbol{V}_x\boldsymbol{B}\boldsymbol{w}\\ &= \tfrac{1}{2}\boldsymbol{x}^T\left(\dot{\boldsymbol{P}} + \boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}\right)\boldsymbol{x}\\ &\quad - \boldsymbol{x}^T\boldsymbol{P}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{P}\boldsymbol{B}\boldsymbol{w}\end{aligned}$$

where $\boldsymbol{u} = -\boldsymbol{\tau} = -\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{x}$. Here, if the above equation is rearranged by using the HJI (16) and Young's inequality $\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{B}\boldsymbol{w} \leq (1/\gamma^2)|\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{B}|^2 + \gamma^2|\boldsymbol{w}|^2$, then we get the following, similar to (6):

$$\dot{V} \leq -\tfrac{1}{2}\boldsymbol{x}^T(\boldsymbol{Q} + \boldsymbol{P}\boldsymbol{B}\boldsymbol{K}\boldsymbol{B}^T\boldsymbol{P})\boldsymbol{x} + \gamma^2|\boldsymbol{w}|^2. \tag{22}$$

Since the right-hand side of inequality (22) is an unbounded function for $\boldsymbol{x}$ and $\boldsymbol{w}$, respectively, hence, the Lagrangian system with an inverse optimal PID controller is an extended disturbance ISS. ■

Corollary 1: The inverse optimal PID controller of (21) exists if and only if the Lagrangian system (4) is extended disturbance ISS.

Proof: The proof consists of Theorems 1–3 and Remark 3 as follows:

$$\begin{array}{ccc}\text{ISS} & \xrightarrow{\text{Th.1}} & \text{Control Law (9)}\\ \text{Th.3}\uparrow & & \downarrow\text{Th.2}\\ \text{Inverse Optimal PID (21)} & \xleftarrow[\text{Re.3}]{} & \mathcal{H}_\infty\ \text{Optimality.}\end{array}$$

■

By Corollary 1, we showed the necessary and sufficient condition for the existence of an inverse optimal PID controller. Though the inverse optimal PID controller guarantees the ISS, it does not give the global asymptotic stability (GAS). This fact brings the performance limitation of an inverse optimal PID controller.

A. Performance Limitation

The extended disturbance of (2) can be expressed as a function of the time and state vector as follows:

$$\boldsymbol{w}(\boldsymbol{x}, t) = \boldsymbol{H}(\boldsymbol{x}, t)\boldsymbol{x} + \boldsymbol{h}(\boldsymbol{x}, t) \tag{23}$$

where

$$\begin{aligned}\boldsymbol{H}(\boldsymbol{x}, t) &= [\boldsymbol{C}\boldsymbol{K}_I, \boldsymbol{M}\boldsymbol{K}_I + \boldsymbol{C}\boldsymbol{K}_P, \boldsymbol{M}\boldsymbol{K}_P]\\ \boldsymbol{h}(\boldsymbol{x}, t) &= \boldsymbol{M}\ddot{\boldsymbol{q}}_d + \boldsymbol{C}\dot{\boldsymbol{q}}_d + \boldsymbol{g} + \boldsymbol{d}.\end{aligned}$$

Now, consider the Euclidian norm of extended disturbance of (23). Then we get the insight such that the extended disturbance can be bounded by the function of the Euclidian norm of a state vector under the following two assumptions:

A1) the configuration derivative $\dot{\boldsymbol{q}}$ is bounded
A2) the external disturbance $\boldsymbol{d}(t)$ is bounded.

The first assumption is not difficult to satisfy if the applied controller can stabilize the system. Also, we think that the second assumption is a minimal information for the unknown external disturbance. By the bounds of $\dot{\boldsymbol{q}}$, the Coriolis and centrifugal matrix $\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ can be bounded, e.g., $|\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})| \leq c_0|\dot{\boldsymbol{q}}|$ with a positive constant c_0. Additionally, we know that the gravitational torque $\boldsymbol{g}(\boldsymbol{q})$ is bounded if the system stays at the earth, and the Inertia matrix $\boldsymbol{M}(\boldsymbol{q})$ is bounded by its own maximum eigenvalue m, e.g., $|\boldsymbol{M}(\boldsymbol{q})| \leq m$. Also, the desired configurations $(\boldsymbol{q}_d, \dot{\boldsymbol{q}}_d, \ddot{\boldsymbol{q}}_d)$ are specified as the bounded values. Therefore, we can derive the following relationship from the above assumptions:

$$\begin{aligned} |\boldsymbol{w}|^2 &= \boldsymbol{x}^T(\boldsymbol{H}^T\boldsymbol{H})\boldsymbol{x} + 2(\boldsymbol{h}^T\boldsymbol{H})\boldsymbol{x} + (\boldsymbol{h}^T\boldsymbol{h}) \\ &\leq c_1|\boldsymbol{x}|^2 + c_2|\boldsymbol{x}| + c_3 \end{aligned} \tag{24}$$

where c_1, c_2 and c_3 are some positive constants. Under the above assumptions, we know that the Euclidian norm of an extended disturbance can be upper bounded by the function of that of a state vector, conversely, the Euclidian norm of a state vector can be lower bounded by the inverse function of that of an extended disturbance

$$|\boldsymbol{w}| \leq \rho_o^{-1}(|\boldsymbol{x}|) \quad \leftrightarrows \quad \rho_o(|\boldsymbol{w}|) \leq |\boldsymbol{x}|$$

where $\rho_o(|\boldsymbol{w}|) = 0$ for $0 \leq |\boldsymbol{w}| \leq \sqrt{c_3}$ because, when $0 \leq |\boldsymbol{w}| \leq \sqrt{c_3}$, necessarily $\boldsymbol{x} = \boldsymbol{0}$. Also, the constant c_3 of (24) cannot be zero either in the case of a trajectory tracking control or in the presence of the external disturbances and the gravitational torques. Though the function ρ_o must be a continuous, unbounded, and increasing function, ρ_o is not a class $\mathcal{K}_\infty$ function because it is not strictly increasing as shown in Fig. 1.

On the other hand, if there exist no the external disturbances $(\boldsymbol{d}(t) = \boldsymbol{0})$ and the gravity torques $(\boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{0})$, then the GAS can be proved for the set-point regulation control $(\ddot{\boldsymbol{q}}_d = \boldsymbol{0}, \dot{\boldsymbol{q}}_d = \boldsymbol{0})$ because $c_2 = 0$, $c_3 = 0$ and the function ρ_o becomes a class $\mathcal{K}_\infty$. For the first time, the GAS of the set-point regulation PD/PID controller was proved for mechanical systems in [3] and [7]. However, either in the trajectory tracking or in the existence of external disturbance, the static PID controller cannot ensure the GAS without using a dynamic model compensator. This fact brings a performance limitation of the inverse optimal PID controller.

The control performance is determined by the gain values of a controller. Hence, it is important to perceive the relation between the gain values and the errors. This relationship can be found by examining the point that the time derivative of the Lyapunov function is equal to zero. The following Theorem suggests a mathematical expression for the performance limitation measure.

Theorem 4: Let $\boldsymbol{K} = k\boldsymbol{I}$, $\boldsymbol{K}_P = k_P\boldsymbol{I}$ and $\boldsymbol{K}_I = k_I\boldsymbol{I} \in \Re^{n\times n}$. Suppose that $\lambda_{\min}$ is the minimum eigenvalue of the following matrix:

$$\boldsymbol{Q}_K = \boldsymbol{Q} + \boldsymbol{PBKB}^T\boldsymbol{P}$$

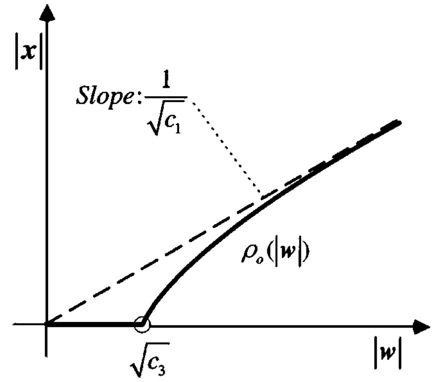

Fig. 1. Function $\rho_o(|\boldsymbol{w}|)$.

and that the performance limitation $|\boldsymbol{x}|_{P.L}$ is defined as the Euclidian norm of a state vector that satisfies $\dot{V} = 0$. If the inverse optimal PID controller in Theorem 3 is applied to the Lagrangian system of (4) and $\lambda_{\min}$ is chosen sufficiently large and γ sufficiently small in order that $\lambda_{\min} - 2\gamma^2 c_1 > 0$ is satisfied, then its performance limitation is upper bounded by

$$|\boldsymbol{x}|_{P.L} \leq \left(\frac{\gamma^2}{\lambda_\gamma}\right)\left[c_2 + \sqrt{c_2^2 + 2c_3\left(\frac{\lambda_\gamma}{\gamma^2}\right)}\right] \tag{25}$$

where c_1, c_2, c_3 are coefficients for the upper bound of the extended disturbance (24), $\lambda_\gamma = \lambda_{\min} - 2\gamma^2 c_1$, and the minimum eigenvalue of $\boldsymbol{Q}_K$ is lower bounded as follows:

$$\lambda_{\min} \geq \min\{k, (k_P^2 - 2k_I)k, k_I^2 k\}. \tag{26}$$

Equation (25) can be regarded as the performance prediction equation which can predict the performance of the closed-loop system as gain changes.

Proof: First, we examine the point that the time derivative of the Lyapunov function (22) stays at zero:

$$\begin{aligned} \dot{V}(\boldsymbol{x}, t) &\leq -\tfrac{1}{2}\,\boldsymbol{x}^T\boldsymbol{Q}_K\boldsymbol{x} + \gamma^2|\boldsymbol{w}|^2 \\ &\leq -\tfrac{1}{2}\,\lambda_\gamma|\boldsymbol{x}|^2 + c_2\gamma^2|\boldsymbol{x}| + c_3\gamma^2 \end{aligned} \tag{27}$$

where $\lambda_\gamma = \lambda_{\min} - 2\gamma^2 c_1$ and the state vector cannot be further reduced at the point satisfying $\dot{V} = 0$. By definition of the performance limitation, the inequality (27) brings the performance limitation of (25). Second, let us consider the minimum eigenvalue $\lambda_{\min}$ of the matrix $\boldsymbol{Q}_K$

$$\begin{aligned} \boldsymbol{Q}_K &= \begin{bmatrix} kk_I^2\boldsymbol{I} & 0 & 0 \\ 0 & k(k_P^2 - 2k_I)\boldsymbol{I} & 0 \\ 0 & 0 & k\boldsymbol{I} \end{bmatrix} \\ &\quad + k\begin{bmatrix} k_I\boldsymbol{I} \\ k_P\boldsymbol{I} \\ \boldsymbol{I} \end{bmatrix}[k_I\boldsymbol{I}, k_P\boldsymbol{I}, \boldsymbol{I}] \\ &= \boldsymbol{Q} + k\boldsymbol{ZZ}^T \end{aligned}$$

where $\boldsymbol{Q}$ is the diagonal positive definite matrix and $\boldsymbol{ZZ}^T$ is a symmetric positive semi-definite matrix. Therefore, the following inequality is always satisfied by Weyl's Theorem in [23] and [24]:

$$\lambda_{\min}(\boldsymbol{Q}) + \lambda_{\min}(k\boldsymbol{ZZ}^T) \leq \lambda_{\min} \triangleq \lambda_{\min}(\boldsymbol{Q}_K). \tag{28}$$

Since $\lambda_{\min}(k\boldsymbol{Z}\boldsymbol{Z}^T)$ is zero, the minimum eigenvalue of $\boldsymbol{Q}_K$ is not smaller than the minimum value among diagonal entries of $\boldsymbol{Q}$. ∎

For the set-point regulation control, the value of the Lyapunov function is large at the start time and it is gradually reduced to zero because the controller is designed so that the time derivative of the Lyapunov function remains negative definite. On the other hand, for the trajectory tracking control, we start the simulation/experiment with zero error $\boldsymbol{x} = \boldsymbol{0}$ after adjusting initial conditions. The value of the Lyapunov function is zero at the start time, however, the error increases to some extent because $\dot{V}(\boldsymbol{0}, 0)$ may have any positive constant smaller than $c_3\gamma^2$ as shown in Fig. 2. This figure depicts the upper bound of $\dot{V}$ versus $|\boldsymbol{x}|$ of the equation (27). Since the performance limitation $|\boldsymbol{x}|_{P.L}$ upper bounded by the inequality (25) is the convergent point as we can see in Fig. 2, the Euclidian norm of a state vector tends to stay at this point. This analysis can naturally illustrate the performance tuning.

B. Performance Tuning

The PID gain tuning has been an important subject; however, until now, it has not been investigated very often. Recently, the noticeable tuning method was suggested as "square law" in [13]. The authors showed that the square law is a good tuning method by their experiments for an industrial robot manipulator. Theoretically, we can confirm once more that the square law is a good tuning method by showing that the performance limitation of (25) can be written approximately as follows:

$$|\boldsymbol{x}|_{P.L} \propto \gamma^2$$

where the square law means that the error is approximately reduced to the square times of the reduction ratio for γ values. Though the square law must be a good performance tuning method, it is not always exact or applicable. The exact performance tuning measure is the performance limitation of (25) in Theorem 4, however, the coefficients c_1, c_2, c_3 are unknowns. To develop the available and more exact tuning method, we rewrite the performance limitation (25) as follows:

$$\begin{aligned} |\boldsymbol{x}|_{P.L} &\le \left(\frac{\gamma}{\sqrt{\lambda_\gamma}}\right)^2 \left[c_2 + \sqrt{c_2^2 + 2c_3\left(\frac{\sqrt{\lambda_\gamma}}{\gamma}\right)^2}\right] \\ &\le \left(\frac{\gamma}{\sqrt{\lambda_\gamma}}\right)^2 \left[2c_2 + \sqrt{2c_3}\left(\frac{\sqrt{\lambda_\gamma}}{\gamma}\right)\right] \\ &= 2c_2\left(\frac{\gamma}{\sqrt{\lambda_\gamma}}\right)^2 + \sqrt{2c_3}\left(\frac{\gamma}{\sqrt{\lambda_\gamma}}\right) \end{aligned} \tag{29}$$

where $\lambda_\gamma = \lambda_{\min} - 2\gamma^2 c_1$. Since the γ value can be chosen to be sufficiently small, we assume that $\lambda_\gamma \approx \lambda_{\min}$. If the values of $k_P^2 - 2k_I$ and k_I are chosen such that their values are bigger than 1, e.g., $k_P^2 - 2k_I > 1$ and $k_I > 1$, then the value of $\lambda_{\min}$ is lower bounded by k, $\lambda_{\min} \ge k$, from (26). By letting $\lambda_\gamma \approx k$ and defining the tuning variable as $(\gamma/\sqrt{k})$, the performance limitation of (29) can be expressed by the function of tuning variable $(\gamma/\sqrt{k})$ as follows:

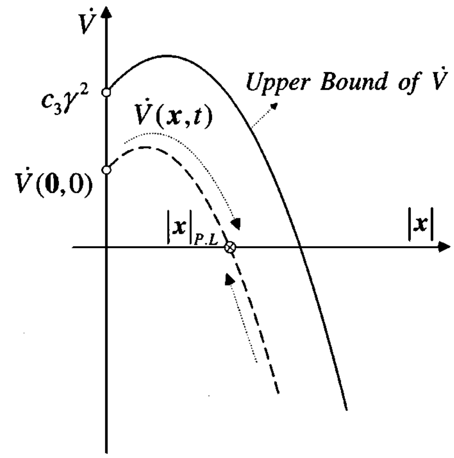

Fig. 2. Relation between $|\boldsymbol{x}|_{P.L}$ and the upper bound of $\dot{V}$.

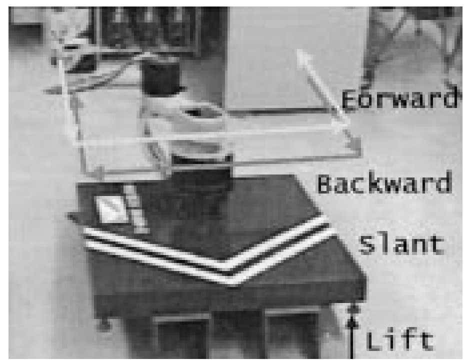

Fig. 3. Desired trajectory of a robot manipulator system.

$$|\boldsymbol{x}|_{P.L} \propto 2c_2\left(\frac{\gamma}{\sqrt{k}}\right)^2 + \sqrt{2c_3}\left(\frac{\gamma}{\sqrt{k}}\right). \tag{30}$$

For a large tuning variable, since the second-order term governs the inequality (30), the following square tuning law is approximately obtained from (30):

$$|\boldsymbol{x}|_{P.L} \propto \gamma^2, \qquad \text{for a small } \sqrt{k}. \tag{31}$$

Also, if the tuning variable $(\gamma/\sqrt{k})$ is small, then we can perceive another linear tuning law

$$|\boldsymbol{x}|_{P.L} \propto \gamma, \qquad \text{for a large } \sqrt{k} \tag{32}$$

because the first-order term of (30) becomes dominant. Here, we propose two tuning methods: one is the coarse tuning which brings the square relation of (31) and the other is the fine tuning which brings the linear relation of (32). Roughly speaking, the coarse tuning is achieved for a small k value and the fine tuning for a large k. As a matter of fact, the criterion between small k and large k is dependent on coefficients c_1, c_2, and c_3; in other words, what k value can be a criterion for a given system should be determined through experiments/simulations.

C. Tuning Variable $(\gamma/\sqrt{k})$ and Gains (k_P and k_I)

In the previous section, although the coarse and fine tuning laws for overall performances were suggested, the performance will be also dependent on the gains k_P and k_I. In this section,

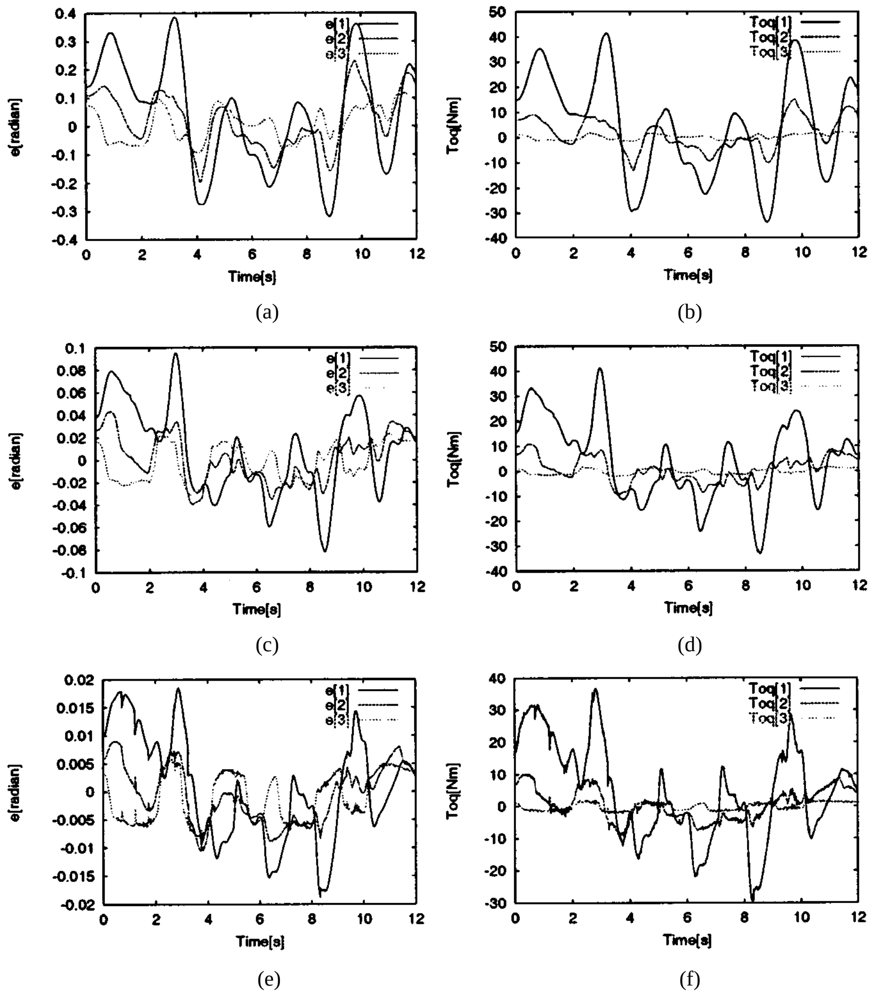

Fig. 4. The control performance for $k = 0.05$, $k_P = 20$ and $k_I = 100$ according to γ change: [square (coarse) tuning rule]. (a) Error $(\gamma = 1)$. (b) Torque $(\gamma = 1)$. (c) Error $(\gamma = 0.5)$. (d) Torque $(\gamma = 0.5)$. (e) Error $(\gamma = 0.25)$. (f) Torque $(\gamma = 0.25)$.

we will suggest upper bounds for gains k_P and k_I based on the time derivative of the Lyapunov function. To begin with, let us reconsider the extended disturbance of (2) as follows:

$$\begin{aligned} |\boldsymbol{w}|^2 = &\left| \boldsymbol{M}\left(\ddot{\boldsymbol{q}}_d + \boldsymbol{K}_P\dot{\boldsymbol{e}} + \boldsymbol{K}_I\boldsymbol{e}\right) \right. \\ &\left. + \boldsymbol{C}\left(\dot{\boldsymbol{q}}_d + \boldsymbol{K}_P\boldsymbol{e} + \boldsymbol{K}_I\int \boldsymbol{e}\right) + \boldsymbol{g} + \boldsymbol{d}\right|^2 \\ =& |\boldsymbol{M}\boldsymbol{K}_P\dot{\boldsymbol{e}} - \boldsymbol{C}\dot{\boldsymbol{e}} + \boldsymbol{M}\boldsymbol{K}_I\boldsymbol{e} + \boldsymbol{C}\boldsymbol{s} + \boldsymbol{h}|^2 \\ \le& 5|\boldsymbol{M}\boldsymbol{K}_P\dot{\boldsymbol{e}}|^2 + 5|\boldsymbol{C}\dot{\boldsymbol{e}}|^2 + 5|\boldsymbol{M}\boldsymbol{K}_I\boldsymbol{e}|^2 \\ &+ 5|\boldsymbol{C}\boldsymbol{s}|^2 + 5|\boldsymbol{h}|^2 \qquad \text{(by Schwarz inequality)} \\ \le& 5(m^2k_P^2 + c_0^2|\dot{\boldsymbol{q}}|^2)|\dot{\boldsymbol{e}}|^2 + 5m^2k_I^2|\boldsymbol{e}|^2 \\ &+ 5c_0^2|\dot{\boldsymbol{q}}|^2|\boldsymbol{s}|^2 + 5|\boldsymbol{h}|^2 \end{aligned} \tag{33}$$

where $\boldsymbol{h} = \boldsymbol{M}\ddot{\boldsymbol{q}}_d + \boldsymbol{C}\dot{\boldsymbol{q}}_d + \boldsymbol{g} + \boldsymbol{d}$, m is a maximum eigenvalue of the Inertia matrix and c_0 the proportional constant of the Coriolis and centrifugal matrix. The following Theorem proposes the selection guidelines for a tuning variable $(\gamma/\sqrt{k})$, gains k_P and k_I using (33).

Theorem 5: Let $|\boldsymbol{M}| \le m$, $|\boldsymbol{C}| \le c_0|\dot{\boldsymbol{q}}|$, $\boldsymbol{K} = k\boldsymbol{I}$, $\boldsymbol{K}_P = k_P\boldsymbol{I}$ and $\boldsymbol{K}_I = k_I\boldsymbol{I} \in \Re^{n\times n}$. Suppose that the tuning variable $(\gamma/\sqrt{k})$ satisfies the following condition:

$$\sqrt{k}/\gamma > \sqrt{10}c_0|\dot{\boldsymbol{q}}| \tag{34}$$

then the gain k_P should be confined to the following constraint:

$$k_P < \frac{\sqrt{(k/\gamma^2) - 10c_0^2|\dot{\boldsymbol{q}}|^2}}{\sqrt{10}m} \tag{35}$$

and the gain k_I should be confined to the following constraint:

$$k_I < \frac{\sqrt{(k/\gamma^2)^2 + 10m^2k_P^2(k/\gamma^2)} - k/\gamma^2}{10m^2}. \tag{36}$$

Proof: If we rewrite the time derivative of the Lyapunov function (22) using (33), then the $\dot{V}$ is obtained as follows:

$$\begin{aligned} \dot{V} \le& -\frac{1}{2}\left(k|\dot{\boldsymbol{e}}|^2 + k(k_P^2 - 2k_I)|\boldsymbol{e}|^2 + kk_I^2\left|\int \boldsymbol{e}\right|^2\right) \\ &- \frac{1}{2}k|\boldsymbol{s}|^2 + \gamma^2|\boldsymbol{w}|^2 \\ \le& -\frac{1}{2}(k - 10\gamma^2(m^2k_P^2 + c_0^2|\dot{\boldsymbol{q}}|^2))|\dot{\boldsymbol{e}}|^2 \\ &- \frac{1}{2}(k(k_P^2 - 2k_I) - 10\gamma^2m^2k_I^2)|\boldsymbol{e}|^2 \\ &- \frac{1}{2}kk_I^2\left|\int \boldsymbol{e}\right|^2 - \frac{1}{2}(k - 10\gamma^2c_0^2|\dot{\boldsymbol{q}}|^2)|\boldsymbol{s}|^2 \\ &+ 5\gamma^2|\boldsymbol{h}|^2. \end{aligned} \tag{37}$$

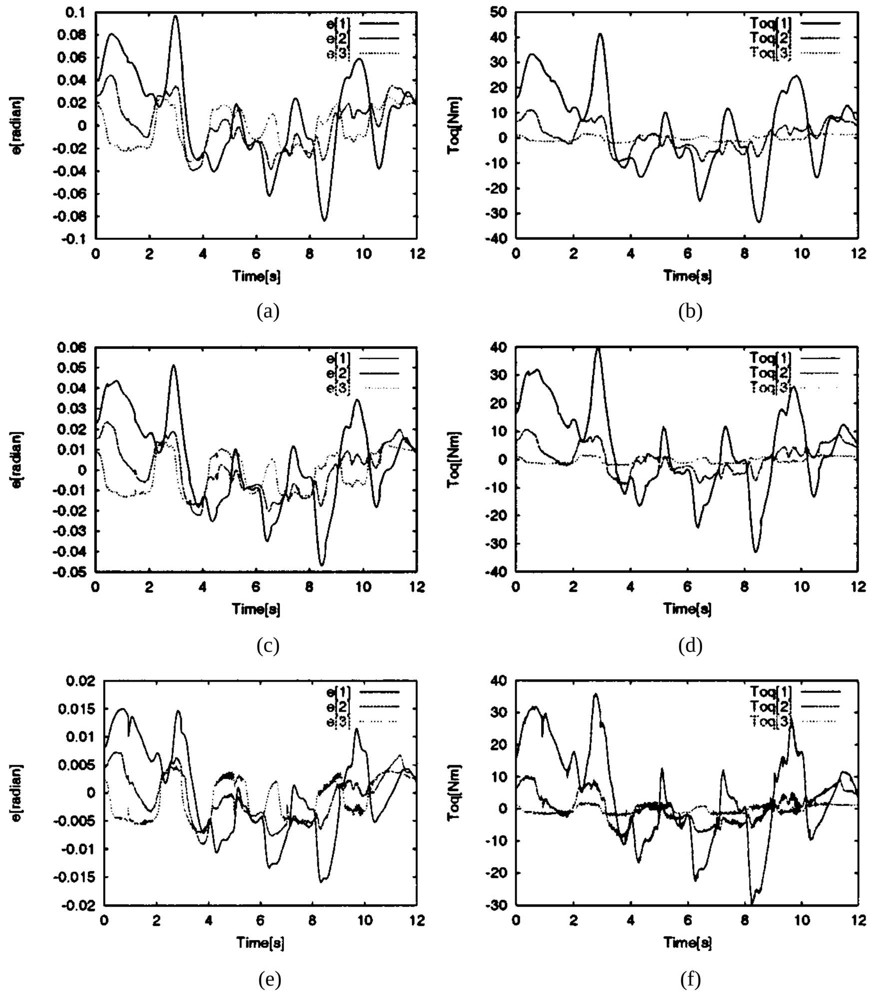

Fig. 5. The control performance for $k = 3, k_P = 20$, and $k_I = 100$ according to γ change: [linear (fine) tuning rule]. (a) Error $(\gamma = 1)$. (b) Torque $(\gamma = 1)$. (c) Error $(\gamma = 0.5)$. (d) Torque $(\gamma = 0.5)$. (e) Error $(\gamma = 0.25)$. (f) Torque $(\gamma = 0.25)$.

If the condition (34) for a tuning variable is satisfied, then the negative definiteness of a term $|\boldsymbol{s}|^2$ in (37) is assured. Also, we can obtain the upper bounds of gains k_P of (35) and k_I of (36) using the negative definiteness of each term $|\dot{\boldsymbol{e}}|^2$ and $|\boldsymbol{e}|^2$, respectively. ∎

Condition (34) in Theorem 5 means that the inverse of the tuning variable should be chosen to be proportional to $|\dot{\boldsymbol{q}}|$, but we cannot know $\dot{\boldsymbol{q}}$ before the experiment. However, the desired configuration derivative $\dot{\boldsymbol{q}}_d$ can be approximately utilized instead of $\dot{\boldsymbol{q}}$. As the maximum $\dot{\boldsymbol{q}}_d$ is faster, the inverse of the tuning variable should be larger

$$\sqrt{k}/\gamma \propto \max\{|\dot{\boldsymbol{q}}_d|\}. \tag{38}$$

Also, condition (35) says that the value of k_P should be determined to be inversely proportional to the maximum eigenvalue of the Inertia matrix. Hence, the large k_P can be used for the small Inertia systems and *vice versa*. In general, since an inverse of tuning variable $(\sqrt{k}/\gamma)$ will be chosen to be large according to (34), we can notice the proportional relation of

$$k_P \propto (1/m)(\sqrt{k}/\gamma) \tag{39}$$

from condition (35). Also, the proportional relation for k_I can be found from (36). If we manipulate (36) using $\sqrt{a^2+b^2} \le |a| + |b|$ as follows:

$$\begin{aligned} k_I &< \frac{\sqrt{(k/\gamma^2)^2 + 10m^2k_P^2(k/\gamma^2)} - k/\gamma^2}{10m^2} \\ &< \frac{(k/\gamma^2) + \sqrt{10}mk_P(\sqrt{k}/\gamma) - (k/\gamma^2)}{10m^2} \\ &= \frac{k_P(\sqrt{k}/\gamma)}{\sqrt{10}m}, \\ k_I &\propto (k_P/m)(\sqrt{k}/\gamma) \end{aligned} \tag{40}$$

then the proportional relation of (40) can be found. Hence, the value of k_I should be chosen to be inversely proportional to m and directly proportional to k_P.

Remark 4: Assume that the Lagrangian system (4) is extended disturbance ISS and the configuration velocity is bounded by applying an inverse optimal PID controller, e.g., $|\dot{\boldsymbol{q}}| < c_4$. If no gravity torque exists ($\boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{0}$), then the term $|\boldsymbol{h}|^2$ in (37) can be upper bounded using the Schwarz inequality as follows:

$$\begin{aligned} |\boldsymbol{h}|^2 &= |\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}}_d + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}_d + \boldsymbol{d}|^2 \\ &\le 3m^2|\ddot{\boldsymbol{q}}_d|^2 + 3c_0^2c_4^2|\dot{\boldsymbol{q}}_d|^2 + 3|\boldsymbol{d}|^2. \end{aligned}$$

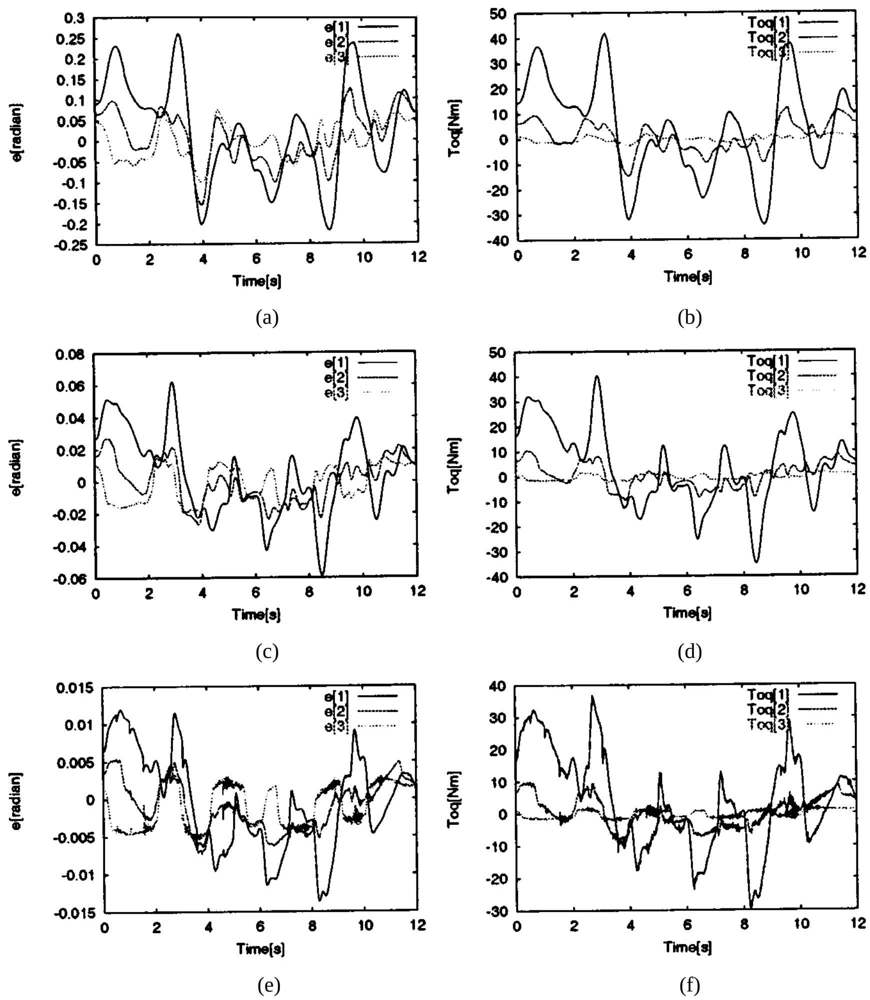

Fig. 6. The control performance for $k = 0.05$, $k_P = 30$ and $k_I = 150$ according to γ change: [square(coarse) tuning rule]. (a) Error ($\gamma = 1$). (b) Torque ($\gamma = 1$). (c) Error ($\gamma = 0.5$). (d) Torque ($\gamma = 0.5$). (e) Error ($\gamma = 0.25$). (f) Torque ($\gamma = 0.25$).

Here, if the above inequality is inserted into (37), then (37) is modified to

$$\begin{aligned}\dot{V} \leq & -\frac{1}{2}(k - 10\gamma^2(2m^2k_P^2 + c_0^2c_4^2))|\dot{\boldsymbol{e}}|^2 \\ & - \frac{1}{2}(k(k_P^2 - 2k_I) - 10\gamma^2m^2k_I^2)|\boldsymbol{e}|^2 \\ & - \tfrac{1}{2}\,kk_I^2\left|\int \boldsymbol{e}\right|^2 - \frac{1}{2}(k - 10\gamma^2c_0^2c_4^2)|\boldsymbol{s}|^2 \\ & + 15\gamma^2m^2|\ddot{\boldsymbol{q}}_d|^2 + 15\gamma^2c_0^2c_4^2|\dot{\boldsymbol{q}}_d|^2 \\ & + 15\gamma^2|\boldsymbol{d}|^2. \end{aligned} \tag{41}$$

Hence, we can mention that it is the reference and external disturbance input ($\dot{\boldsymbol{q}}_d$, $\ddot{\boldsymbol{q}}_d$, $\boldsymbol{d}$)-to-state ($\boldsymbol{e}$, $\dot{\boldsymbol{e}}$, $\int \boldsymbol{e}$, $\boldsymbol{s}$) stable from (41), additionally.

V. Experimental Results

To show the validity of the tuning rules of an inverse optimal PID controller experimentally, we employ a planar three-link robot manipulator as a typical example of a Lagrangian system. Additionally, we inclined the planar robot in Fig. 3 about 10° to give the gravity effect. This mechanical manipulator is composed of three direct drive motors: the base motor has 200-Nm capability, the second one 55 Nm, and the third one 18 Nm. To make use of the unified control scheme, we scaled up the applied torque according to the capacity of motor, e.g., the scaling factor of the base motor is 8.0, that of the second one is 3.0, and that of the third one is 1.0. This robot system interfaces with a computer via the interface boards such as the encoder counting PCI board (S626 manufactured by Sensoray Co.) and D/A converting ISA board (PCL726 manufactured by Advantech Co.). The RTLinux V2.2 is used as the real-time operating system of the computer. We can raise the control frequency up to 5 kHz thanks to the RTLinux. The desired trajectories are the fifth-order polynomial functions of time so that the initial/final velocity and acceleration can be set to zeros. As shown in Fig. 3, they are composed of three forward line segments and three backward ones; also, the execution time per each line segment is 2 s and the total execution time is 12 s.

Here, the inverse optimal PID controller has the following form:

$$\boldsymbol{\tau} = \left(k + \frac{1}{\gamma^2}\right)\left(\dot{\boldsymbol{e}} + k_P\boldsymbol{e} + k_I\int \boldsymbol{e}\,dt\right)$$

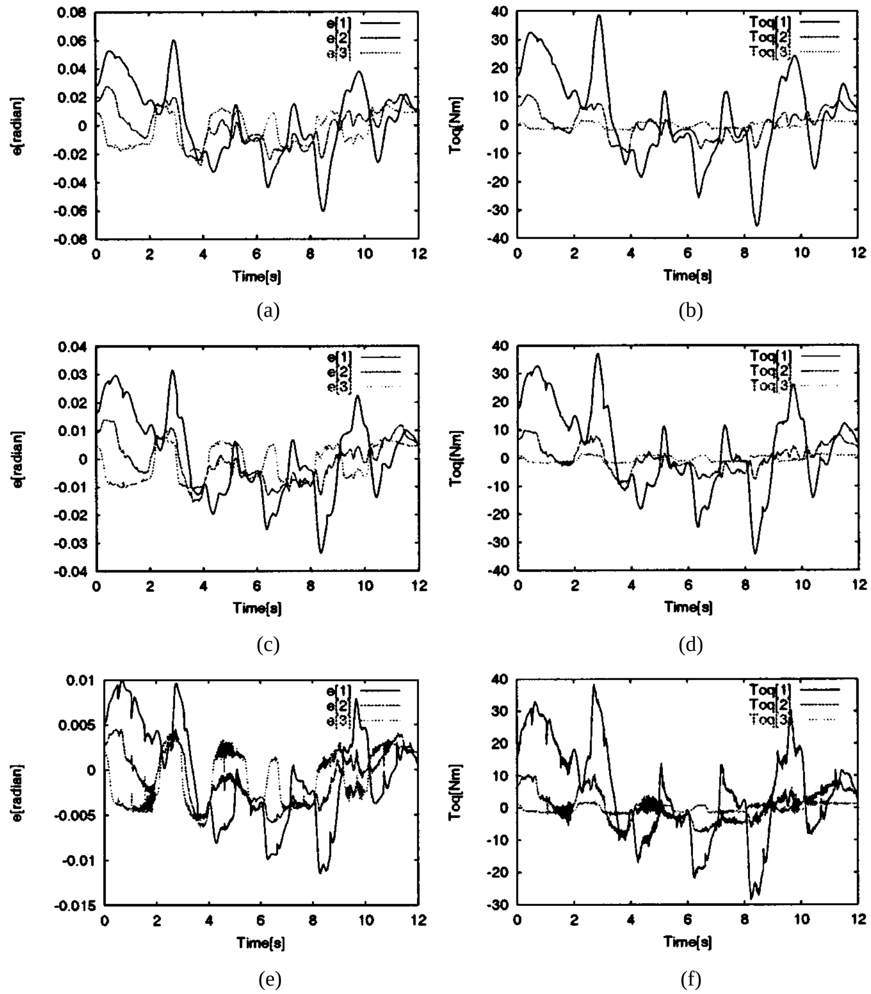

Fig. 7. The control performance for $k = 3$, $k_P = 30$ and $k_I = 150$ according to γ change: [linear(fine) tuning rule]. (a) Error $(\gamma = 1)$. (b) Torque $(\gamma = 1)$. (c) Error $(\gamma = 0.5)$. (d) Torque $(\gamma = 0.5)$. (e) Error $(\gamma = 0.25)$. (f) Torque $(\gamma = 0.25)$.

and, as proved in Theorem 2 and Remark 3, it is optimal for the $\mathcal{H}_\infty$ performance index of (15) using

$$\boldsymbol{Q} = \begin{bmatrix} k_I^2 k\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & (k_P^2 - 2k_I)k\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & k\boldsymbol{I} \end{bmatrix} \tag{42}$$

$$\boldsymbol{R} = \left(k + \frac{1}{\gamma^2}\right)^{-1} \boldsymbol{I}. \tag{43}$$

At first, we determined initial gains $k_P = 20$ and $k_I = 100$ that satisfy conditions 1 and 2 in Theorem 3. After fixing $k = 0.05$ and $\gamma = 1.0$, we obtained the configuration error ($\boldsymbol{e}$) profile of Fig. 4(a) and the applied torque ($\boldsymbol{\tau}$) profile of Fig. 4(b). Also, we obtained Fig. 4(c) and (d) for $\gamma = 0.5$ and Fig. 4(e) and (f) for $\gamma = 0.25$. In Fig. 4, whenever the γ values are halved like $1.0 \rightarrow 0.5 \rightarrow 0.25$, the maximum configuration errors are approximately reduced to a quarter like $0.4 \rightarrow 0.1 \rightarrow 0.02$. Hence, it complied with the "square (coarse) tuning rule." Second, we increased the k value gradually and, at $k = 3$, the results shown in Fig. 5 were obtained by the same experiments. Whenever the γ values are halved for $k = 3$, the errors are approximately reduced to a half like $0.1 \rightarrow 0.05 \rightarrow 0.016$. Hence, it corresponded to the "linear (fine) tuning rule." The torques were not changed much as we can see in Figs. 4 and 5.

In the above experiments, we showed the validity of the tuning laws for fixed k_P and k_I. As a matter of fact, the change of gains k_P and k_I also affects on the performance of the control system. To begin with, we should remember two proportional relations of (39) and (40). If the gain k_P is increased to one and a half times (k_P: $20 \rightarrow 30$), then the gain k_I should be increased by the same incremental ratio (k_I: $100 \rightarrow 150$) according to a proportional relation (40). Although we had changed gains to $k_P = 30$ and $k_I = 150$, the experimental results for $k = 0.05$ complied with the square tuning rule, e.g., $0.25 \rightarrow 0.06 \rightarrow 0.014$ as shown in Fig. 6 and those of $k = 3$ complied with the linear tuning rule, e.g., $0.06 \rightarrow 0.03 \rightarrow 0.011$ as shown in Fig. 7.

Additionally, the performance change by the increment of k_P and k_I can be estimated through the state-weighting matrix of (42). For example, if we increase gains k_P and k_I by two times, then the magnitude of (42) is increased approximately by the square of two times. Therefore, the state vector $\boldsymbol{x}$ will be reduced to a half, in other words, the control performance becomes better by two times. In our experiments, since the increment ratio was one and a half times, we could see the performance enhanced by one and a half times. Comparing Figs. 4 and 6, then the performances for the error were enhanced by one and a half times, e.g., $0.4 \rightarrow 0.25$ in (a), $0.1 \rightarrow 0.06$ in (c), and $0.02 \rightarrow 0.014$ in

TABLE I
THE EXPERIMENTAL RESULTS FOR VARIOUS γ AND k VALUES, WHERE THE SUBSCRIPT u OR l MEANS THE CORRESPONDING VALUE OF EITHER UPPER OR LOWER DATA

k_P/k_I	k	γ	$\\|x\\|$	$\\|x\\|_u/\\|x\\|_l$	Expected
20/100	0.05	1	2.147	3.890	4.0
		0.5	0.552	3.101	
		0.25	0.178		
	0.5	1	1.516	3.050	≀
		0.5	0.497	2.873	
		0.25	0.173		
	1	1	1.122	2.499	
		0.5	0.449	2.641	
		0.25	0.170		
	3	1	0.562	1.698	2.0
		0.5	0.331	2.006	
		0.25	0.165		
30/150	0.05	1	1.545	3.796	4.0
		0.5	0.407	2.576	
		0.25	0.158		
	0.5	1	1.072	2.905	≀
		0.5	0.369	2.306	
		0.25	0.160		
	1	1	0.806	2.371	
		0.5	0.340	2.112	
		0.25	0.161		
	3	1	0.409	1.623	2.0
		0.5	0.252	1.565	
		0.25	0.161		

(e). Also, we could see the same performance enhancement in Figs. 5 and 7. From these experimental results, we could confirm experimentally that the inverse optimal PID controller is optimal for the $\mathcal{H}_\infty$ performance index of (15).

We showed through experiments that the maximum configuration errors complied with the square/linear tuning rules. However, it does not imply that the average control performances comply with the tuning rules. Now, we consider the $\mathcal{L}_2$ norm performance for a state vector. In Table I, the $\mathcal{L}_2$ norm performances were evaluated by

$$\|x\| = \sqrt{\int_0^{12} \left[\dot{e}^T\dot{e} + e^Te + \int e^T \int e\right] dt}.$$

Also, for $k = 0.5$ and $k = 1$, the same experiments were also performed. The average performance for the $k = 0.05$ complied with the square tuning rule and for $k = 3$ the linear one as shown in Table I. For the intermediate values such as $k = 0.5$ and $k = 1$, the average performances showed the intermediate performance enhancement between the square and linear tuning. As a matter of fact, the performance enhancement was governed by (30) for an intermediate k value. For the $k_P = 20$, $k_I = 100$, the maximum deviation for the square/linear tuning rules between the experimental and the expected value was 22.5% in the upper part of Table I. Also, for $k_P = 30$, $k_I = 150$, the maximum deviation became 35.6% in the lower part of Table I.

VI. CONCLUDING REMARKS

The existence of an inverse optimal PID controller was proved using the extended disturbance ISS. It was found to be optimal for the $\mathcal{H}_\infty$ performance index under certain PID gain conditions. Also, a performance analysis of the inverse optimal PID controller was provided in view of the performance limitation and tuning. The simple coarse/fine performance tuning rules were derived from the performance limitation measures. Selection guidelines were suggested for PID gains, and the effect of gain variations on the control performance was analyzed. The experimental results demonstrated the validity of this analysis on the performance tuning laws and $\mathcal{H}_\infty$ optimality.

REFERENCES

[1] K. Astrom and T. Hagglund, *PID Controllers: Theory, Design, and Tuning*. Research Triangle Park, NC: Instrument Soc. Amer., 1995.

[2] C. C. Yu, *Autotuning of PID Controllers: Relay Feedback Approach*. New York: Springer-Verlag, 1999.

[3] S. Arimoto and F. Miyazaki, "Stability and robustness of PID feedback control for robot manipulators of sensory capability," in *Robot. Res.*, M. Brady and R. Paul, Eds., 1984, pp. 783–799.

[4] R. Kelly, "Global positioning of robot manipulators via PD control plus a class of nonlinear integral actions," *IEEE Trans. Automat. Contr.*, vol. 43, pp. 934–937, July 1998.

[5] R. Ortega, A. Loria, and R. Kelly, "A semi-globally stable output feedback PI^2D regulator for robot manipulators," *IEEE Trans. Automat. Contr.*, vol. 40, pp. 1432–1436, Aug. 1995.

[6] J. A. Ramirez, I. Cervantes, and R. Kelly, "PID regulation of robot manipulators: Stability and performance," *Syst. Contr. Lett.*, vol. 41, pp. 73–83, 2000.

[7] M. Takegaki and S. Arimoto, "A new feedback method for dynamic control of manipulators," *Trans. ASME J. Dyn. Syst., Meas. Contr.*, vol. 102, pp. 119–125, 1981.

[8] A. Isidori and A. Astolfi, "Disturbance attenuation and $\mathcal{H}_\infty$-control via measurement feedback in nonlinear systems," *IEEE Trans. Automat. Contr.*, vol. 37, pp. 1283–1293, 1992.

[9] A. J. Vander Schaft, "$\mathcal{L}_2$-gain analysis of nonlinear systems and nonlinear state feedback $\mathcal{H}_\infty$ control," *IEEE Trans. Automat. Contr.*, vol. 37, pp. 770–784, June 1992.

[10] Y. Choi, W. K. Chung, and Y. Youm, "Robust control of manipulators using Hamiltonian optimization," in *Proc. IEEE Int. Conf. Robotics and Automation*, 1997, pp. 2538–2364.

[11] D. Dawson, M. Grabbe, and F. L. Lewis, "Optimal control of a modified computed-torque controller for a robotic manipulator," *Int. J. Robot. Automat.*, vol. 6, pp. 161–165, 1991.

[12] J. Park and W. K. Chung, "Analytic nonlinear $\mathcal{H}_\infty$ inverse-optimal control for Euler–Lagrange system," *IEEE Trans. Robot. Automat.*, vol. 16, pp. 847–854, 2000.

[13] J. Park and W. K. Chung, "Design of a robust $\mathcal{H}_\infty$ PID control for industrial manipulators," *Trans. ASME J. Dyn. Syst., Meas. Contr.*, pp. 803–812, 2000.

[14] J. Park, W. K. Chung, and Y. Youm, "Analytic nonlinear $\mathcal{H}_\infty$ optimal control for robot manipulators," in *Proc. IEEE Int. Conf. Robotics and Automation*, 1998, pp. 2709–2715.

[15] W. Su, C. E. de Souza, and L. Xie, "$\mathcal{H}_\infty$ control for asymptotically stable nonlinear systems," *IEEE Trans. Automat. Contr.*, vol. 44, pp. 989–993, May 1999.

[16] D. Angeli, "Input-to-state stability of PD-controlled robotic systems," *Automatica*, vol. 35, pp. 1285–1290, 1999.

[17] Y. Choi and W. K. Chung, "On the optimality and performance of PID controller for robotic manipulators," in *Proc. IEEE Int. Conf. Robotics and Automation*, 2001, pp. 1142–1148.

[18] M. Krstic, I. Kanellakopoulos, and P. Kokotovic, *Nonlinear and Adaptive Control Design*. New York: John Wiley, 1995.

[19] E. D. Sontag, "Smooth stabilization implies coprime factorization," *IEEE Trans. Automat. Contr.*, vol. 34, pp. 435–443, Apr. 1989.

[20] E. D. Sontag and Y. Wang, "On characterizations of the input-to-state stability property," *Syst. Contr. Lett.*, vol. 24, pp. 351–359, 1995.

[21] M. Krstic and Z. H. Li, "Inverse optimal design of input-to-state stabilizing nonlinear controllers," *IEEE Trans. Automat. Contr.*, vol. 43, pp. 336–350, Mar. 1998.
[22] M. Krstic and P. Tsiotras, "Inverse optimal stabilization of a rigid spacecraft," *IEEE Trans. Automat. Contr.*, vol. 44, pp. 1042–1049, May 1999.
[23] G. H. Golub and C. F. Van Loan, *Matrix Computations*, 3rd ed. Baltimore, MD: The Johns Hopkins Univ. Press, 1996.
[24] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge, MA: Cambridge Univ. Press, 1985.

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