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Robust adaptive synchronization of dynamical networks via scalar controllers

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ABSTRACT

Based on high gain feedback control theory, robust adaptive synchronization of dynamical network is investigated in this paper. When the non-linear coupling functions are unknown but with unknown bounded, some fairly simple robust adaptive scalar feedback controllers are derived. The key idea is that a time-varying gain parameter is introduced in designing controllers which can guarantee that the states of uncertain coupled dynamical networks robust adaptive asymptotically synchronize with each other. Numerical simulation is given to validate the proposed theoretical result.

1. Introduction

Chaos synchronization has been studied extensively in recent years. Much effort has been devoted to investigate it and numerous achievements [1–8,17] have been made. It is well known that in communication and control systems, one would like to minimize the number of signals to be sent and the number of controllers to be designed. Therefore being able to synchronize with a scalar signal is beneficial to the design of chaotic communication and control systems. Recently, synchronization of complex dynamical networks, such as small-world and scale-free dynamical networks has been studied extensively. One of the topics is on large-scale and complex networks of chaotic oscillators [1–5]. In most of these studies, a common approach is to linearize non-linear dynamical nodes around the synchronized state of the network thereby obtaining some *local* synchronization and stability results. More recently, a sufficient condition for global synchronization was derived in Ref. [9] based on Lyapunov stability theory and introducing a reference state trajectory. Unlike other approaches where only local results are obtained, the complex network is not linearized in this case.

However, in these studies, the final state that the networks reached after achieving synchronization is usually unknown beforehand. To deal with this problem, we recently presented a kind of control method in Ref. [10] such that the states of the complex network synchronize to a desired orbit. However, in most existing results, the number of controllers is not addressed. In particular, it would be desirable that the coupled chaotic network can be synchronized via some scalar controllers since these minimize the number of controllers that need to be designed [1]. For solving the above-described

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problem, we proposed some fairly simple scalar state feedback controllers by exploiting the high gain state feedback control theory in Ref. [11]. However, the coupling functions are assumed with known bounded in Ref. [11]. If the bounds of coupling functions are unknown, the proposed control method in Ref. [11] will fail in guarantee that the states of network synchronize each other. In this paper, when the non-linear coupling functions of the network are unknown but with unknown bounded, some fairly simple robust adaptive scalar feedback controllers are indeed derived. The key idea is that a time-varying gain parameter is introduced in the designing of the scalar controllers which can make the states of uncertain coupled chaotic dynamical networks robust adaptive asymptotically synchronize with each other. A typical example of complex network with chaotic nodes is finally used to verify the theoretical results and the effectiveness of the proposed scalar controllers.

Throughout this paper, we denote $\|x\|$ as the vector 2-norm of the vector x, denote $\|P\|$ as the 2-norm of the matrix P induced by the vector 2-norm. $A^{\mathrm{T}}(x^{\mathrm{T}})$ means the transpose of the matrix A (or vector x). The absolute values of the real numbers $a \in R$ is denoted by $|a|$.

2. Problem formulation

Consider the dynamical network which consists of N coupled nodes with non-linear uncertain chaos system

$$\begin{cases} \dot{x}_{i1} = x_{i2} + f_1\left(x_{i1}, x_{11}, x_{21}, \ldots, x_{N1}, t\right) \\ \dot{x}_{i2} = x_{i3} + f_2\left(x_{i1}, x_{i2}, x_{12}, x_{22}, \ldots, x_{N2}, t\right) \\ \vdots \\ \dot{x}_{in} = u_i + f_n\left(x_i, x_{1n}, x_{2n}, \ldots, x_{Nn}, t\right), \end{cases} \tag{1}$$

where the $x_i = (x_{i1}, x_{i2}, \ldots, x_{in}) \in R^n$, $i = 1, 2, \ldots, N$, represents the state vector of the system, the f_i, $i = 1, 2, \ldots, n$, are unknown smooth non-linear functions, and $u_i \in R$, $i = 1, 2, \ldots, N$ are the scalar control inputs to be determined later.

Remark. The model (1) is some extended physical models which exist in broad physical systems, such as Jet engine compression systems [12], active suspension [13], and Chua's chaotic circuit [14].

For example: A Chua's chaotic circuit is described by

$$\begin{aligned} \dot{s}_1 &= p\left(-s_1 + s_2 - \phi\left(s_1\right)\right) \\ \dot{s}_2 &= s_1 - s_2 + s_3 \\ \dot{s}_3 &= -q s_2 \end{aligned}$$

where $\phi(s_1) = m_0 s_1 + \frac{1}{2}\left(m_1 - m_0\right)\left(|s_1 + 1| - |s_1 - 1|\right)$ with $m_0 < 0$ and $m_1 < 0$. For ensuring the chaotic behavior, the parameters are selected as $p = 10$, $q = 14.7$, $m_0 = -0.68$, and $m_1 = -1.27$.

Let $x = \frac{s_1}{p}$, $y = s_2$, $z = s_3$, we have

$$\begin{aligned} \dot{x} &= y - px - \phi\left(px\right) \\ \dot{y} &= z + px - y \\ \dot{z} &= -qy. \end{aligned}$$

We can extend Chua's chaotic circuit a controlled network as follows

$$\dot{x}_{i1} = x_{i2} - px_{i1} - \phi\left(px_{i1}\right) + \sum_{j=i}^{i+2} \tanh\left(x_{j1}\right)$$

$$\begin{aligned} \dot{x}_{i2} &= x_{i3} + px_{i1} - x_{i2} \\ \dot{x}_{i3} &= u_i - qx_{i2}, \end{aligned} \qquad i = 1, 2, \ldots, N-2$$

and

$$\dot{x}_{i1} = x_{i2} - px_{i1} - \phi\left(px_{i1}\right) + \tanh\left(x_{i-1,1}\right) + 2\tanh\left(x_{i1}\right)$$

$$\begin{aligned} \dot{x}_{i2} &= x_{i3} + px_{i1} - x_{i2} \\ \dot{x}_{i3} &= u_i - qx_{i2}, \end{aligned} \qquad i = N-1, N$$

where

$$f_1(x) = -px_{i1} - \phi\left(px_{i1}\right) + \sum_{j=i}^{i+2} \tanh\left(x_{j1}\right), \qquad f_2(x) = px_{i1} - x_{i2}, \qquad f_3(x) = -qx_{i2}, \quad i = 1, 2, \ldots, N-2$$

$$f_1(x) = -px_{i1} - \phi\left(px_{i1}\right) + \tanh\left(x_{i-1,1}\right) + 2\tanh\left(x_{i1}\right),$$

$$f_2(x) = px_{i1} - x_{i2}, \qquad f_3(x) = -qx_{i2}, \quad i = N-1, N.$$

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The objective is to find some smooth scalar controllers $u_i \in R, i = 1, 2, \ldots, N$ such that the solutions of systems (1) synchronize with each other, in the sense that

$$\lim_{t\to\infty} \left\| x_i(t) - x_j(t) \right\| = 0, \quad \text{for all } i, j = 1, 2, \ldots, N \tag{2}$$

Let $e_i = x_i - x_{i+1}, i = 1, 2, \ldots, N-1, e_{ij} = x_{ij} - x_{i+1,j}, i = 1, 2, \ldots, N-1, j = 1, \ldots, n, e_i = (e_{i1}, \ldots, e_{in})^{\mathrm{T}}, e_{ij} \in R$, one can obtain the error dynamical systems

$$\begin{cases} \dot{e}_{i1} = e_{i2} + \tilde{f}_1 \left(x_{i1}, x_{i+1,1}, x_{11}, x_{21}, \ldots, x_{N1}, t \right) \\ \dot{e}_{i2} = e_{i3} + \tilde{f}_2 \left(x_{i1}, x_{i2}, x_{i+1,1}, x_{i+1,2}, x_{12}, x_{22}, \ldots, x_{N2}, t \right) \\ \vdots \\ \dot{e}_{in} = \tilde{u}_i + \tilde{f}_n \left(x_i, x_{i+1}, x_{1n}, x_{2n}, \ldots, x_{Nn}, t \right) \end{cases} \tag{3}$$

where

$$\tilde{f}_k \left(x_{ik}, x_{i+1,k}, t \right) = f_k \left(x_{i1}, x_{i2}, \ldots, x_{ik}; x_{1k}, \ldots, x_{Nk}; t \right) - f_k \left(x_{i+1,1}, x_{i+1,2}, \ldots, x_{i+1,k}; x_{1k}, \ldots, x_{Nk}; t \right)$$
$$\tilde{u}_i = u_i - u_{i+1}, \quad i = 1, 2, \ldots, N-1.$$

For attaining the objective Eq. (2), an assumption is made as follows:

Assumption 1. Let

$$\left| \tilde{f}_k \left(x_{ik}, x_{i+1,k}, t \right) \right| \le \rho_i(t) \sum_{j=1}^{k} \left| e_{ij} \right|, \quad i = 1, 2, \ldots, N-1, \; k = 1, 2, \ldots, n, \tag{4}$$

where $\rho_i(t), i = 1, 2, \ldots, N-1$, are unknown time-varying parameters but non-negative.

Remark. Assumption 1 is similar to the Lipschitz condition. However, Assumption 1 is different from Lipschitz condition. In Assumption 1, we take time-varying parameters $\rho_i(t), i = 1, 2, \ldots, N-1$ instead of Lipschitz constant.

For example, we consider N identical Rössler chaotic systems

$$\dot{x}_{i1} = x_{i2} + a x_{i1},$$
$$\dot{x}_{i2} = x_{i3} - x_{i1},$$
$$\dot{x}_{i3} = b + x_{i3} \left(x_{i2} - c \right), \quad i = 1, 2, \ldots, N.$$

We have

$$\left| \tilde{f}_1 \left(x_{i1}, x_{i+1,1}, t \right) \right| = a \left| e_{i1} \right|, \qquad \left| \tilde{f}_2 \left(x_{i2}, x_{i+1,2}, t \right) \right| = \left| e_{i1} \right|, \qquad \left| \tilde{f}_3 \left(x_{i2}, x_{i+1,2}, t \right) \right| \le \left| x_{i3} \right| \left| e_{i2} \right| + \left(c + \left| x_{i+1,2} \right| \right) \left\| e_{i3} \right\|.$$

From above, we define $\rho_i(t) = \max \left\{ a, 1, |x_{i3}|, \left(c + \left| x_{i+1,2} \right| \right) \right\}, i = 1, 2, \ldots, N-1$, then Assumption 1 holds.

3. Main result

Assumption 2. In Assumption 1, let $\rho_i(t) \in \left[\underline{\rho_i}, \overline{\rho_i} \right], i = 1, 2, \ldots, N-1$, and $\underline{\rho_i}, \overline{\rho_i}$ are unknown positive constants.

Definition 1. The synchronization of dynamical network (1) is said to be robust, if the stability properties of the objective (2) are persistent with respect to the uncertain perturbations of the dynamical network (1) involved.

Definition 2. The synchronization of dynamical network (1) is said to be robust adaptive synchronization, if there exist the adaptive controllers such that the synchronization of dynamical network (1) is robust.

Now, we derive robust adaptive controller for objective (2) as follows:
We denote $\rho = \max_{1 \le i \le N-1} \bar{\rho}_i$ which is also an unknown positive constant and choose the controllers

$$\tilde{u}_i = -\left(L^n(t) a_1 e_{i1} + L^{n-1}(t) a_2 e_{i2} + \cdots + L(t) a_n e_{in} \right), \quad i = 1, 2, \ldots, N-1, \tag{5}$$

where $L(t)$ is a time-varying gain parameter to be determined later, and $a_k > 0, k = 1, 2, \ldots, n$ are coefficients of the Hurwitz polynomial

$$p(s) = s^n + a_n s^{n-1} + a_{n-1} s^{n-2} + \cdots + a_2 s + a_1.$$

By introducing a transformation

$$\varepsilon_{ik} = \frac{e_{ik}}{L^k(t)}, \quad i = 1, 2, \ldots, N-1, \; k = 1, 2, \ldots, n, \tag{6}$$

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the systems (3) are transformed into

$$\begin{cases} \dot{\varepsilon}_{i1} = L(t)\varepsilon_{i2} + \dfrac{\tilde{f}_1\left(x_{i1}, x_{i+1,1}, t\right)}{L^1(t)} - \varepsilon_{i1}\dfrac{\mathrm{d}\ln L^1(t)}{\mathrm{d}t} \\ \dot{\varepsilon}_{i2} = L(t)\varepsilon_{i3} + \dfrac{\tilde{f}_2\left(x_{i1}, x_{i2}, x_{i+1,1}, x_{i+1,2}, t\right)}{L^2(t)} - \varepsilon_{i2}\dfrac{\mathrm{d}\ln L^2(t)}{\mathrm{d}t} \\ \vdots \\ \dot{\varepsilon}_{in} = L(t)\left(-a_1\varepsilon_{i1} - a_2\varepsilon_2 \cdots - a_n\varepsilon_{in}\right) + \dfrac{\tilde{f}_n\left(x_i, x_{i+1}, t\right)}{L^n(t)} - \varepsilon_{in}\dfrac{\mathrm{d}\ln L^n(t)}{\mathrm{d}t}. \end{cases} \tag{7}$$

The dynamical system (7) can be expressed in a matrix differential equation

$$\dot{\varepsilon}_i = \left(L(t)A - \bar{L}(t)\right)\varepsilon_i + F_i(t), \quad i = 1, 2, \ldots, N-1, \tag{8}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_1 & -a_2 & -a_3 & \cdots & -a_n \end{bmatrix}, \qquad \varepsilon_i = \left(\varepsilon_{i1}, \varepsilon_{i2}, \ldots, \varepsilon_{in}\right)^{\mathrm{T}},$$

$$F_i(t) = \left(F_{i1}(t), F_{i2}(t), \ldots, F_{in}(t)\right)^{\mathrm{T}},$$

$$F_{ik}(t) = \frac{\tilde{f}_k\left(x_{i1}, \ldots, x_{ik}, x_{i+1,1}, \ldots, x_{i+1,k}, t\right)}{L^k(t)}, \quad k = 1, 2, \ldots, n,$$

$$\bar{L}(t) = \mathrm{diag}\left(\mathrm{d}\ln L^1(t)/\mathrm{d}t \quad \cdots \quad \mathrm{d}\ln L^n(t)/\mathrm{d}t\right).$$

We can easily prove the matrix A to be a Hurwitz matrix and the Hurwitz polynomial of the matrix A be

$$p(s) = s^n + a_n s^{n-1} + a_{n-1}s^{n-2} + \cdots + a_2 s + a_1.$$

Therefore, there exists positive definite matrix P satisfy $A^{\mathrm{T}}P + PA = -I$. If we define $\hat{M}(t)$ as the estimate of the unknown parameter $M = \rho + \frac{n^2}{4}\alpha^3 D^2$ where $\alpha = 4\sqrt{n}\,\|P\|$, D is the unknown constant which is the bound of the region $B_D = \left\{e : e^{\mathrm{T}}e \leq D, D > 0\right\}$. And we choose an update law for unknown parameter M as follows:

$$\dot{\hat{M}}(t) = 4\sqrt{n}\,\|P\|\,\varepsilon^{\mathrm{T}}\varepsilon,$$

where $\hat{M}(0) > 0$, $\varepsilon = (\varepsilon_1^{\mathrm{T}}, \ldots, \varepsilon_{N-1}^{\mathrm{T}})^{\mathrm{T}}$ and $\|P\|$ denotes the norm of the positive definite matrix P. We have the following closed loop dynamical systems

$$\begin{cases} \dot{\varepsilon}_i = \left(L(t)A - \bar{L}(t)\right)\varepsilon_i + F_i(t) \\ \dot{\hat{M}}(t) = 4\sqrt{n}\,\|P\|\,\varepsilon^{\mathrm{T}}\varepsilon, \end{cases} \quad i = 1, 2, \ldots, N-1 \tag{9}$$

where $L(t) = L_1 + \alpha\hat{M}(t)$, $\tilde{M}(t) = \hat{M}(t) - M$, $L_1 > 2$, $\alpha = 4\sqrt{n}\,\|P\|$.

Now, we can derive the following result:

Theorem 1. *Let Assumptions 1 and 2 hold. Then, for any unknown bounded region $B_D = \left\{e : e^{\mathrm{T}}e \leq D, D > 0\right\}$ in which D is an arbitrary unknown constant and $e = \left(e_1 \quad \cdots \quad e_{N-1}\right)^{\mathrm{T}}$, there exist the scalar robust adaptive controllers*

$$u_i = u_1 + L^n(t)a_1\left(x_{11} - x_{i1}\right) + L^{n-1}(t)a_2\left(x_{12} - x_{i2}\right) + \cdots + L(t)\,a_n\left(x_{1n} - x_{in}\right), \quad i = 2, \ldots, N, \tag{10}$$

with

$$L(t) = L_1 + \alpha\hat{M}(t), \quad L_1 > \max\left\{2, \lambda_{\max}\left(P^2\right)\right\}, \ \alpha = 4\sqrt{n}\,\|P\|,$$

and update law

$$\dot{\hat{M}}(t) = 4\sqrt{n}\,\|P\|\sum_{i=1}^{N-1}\varepsilon_i^{\mathrm{T}}\varepsilon_i, \qquad \hat{M}(0) > 0, \tag{11}$$

such that the solutions of systems (9) at $\varepsilon(t) = 0$, $\tilde{M}(t) = 0$ are uniformly stable on the bounded region B_D. In addition, $\lim_{t\to\infty}\varepsilon_i(t) = 0$, $i = 1, 2, \ldots, N-1$.

Where the time-varying parameter $\hat{M}(t)$ is an estimate for the unknown parameter $M = \rho + n^2\alpha^3 D^2$. Positive constant $\hat{M}(0) > 0$ is the initial value of the time-varying parameter $\hat{M}(t)$.

Now, we give the proof of Theorem 1 as follows:

Proof. Since A is a Hurwitz matrix, there exists positive definite matrix P satisfy

$$A^{\mathrm{T}}P + PA = -I. \tag{12}$$

Consider a Lyapunov function candidate of the form

$$V(t) = \sum_{i=1}^{N-1} \varepsilon_i^{\mathrm{T}} P \varepsilon_i + \frac{\tilde{M}^2(t)}{2} \tag{13}$$

where $\tilde{M}(t) = \hat{M}(t) - M$.

The time derivative of $V(t)$ along the trajectory of the system (9) is given by

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} \leq -L(t)\sum_{i=1}^{N-1} \varepsilon_i^{\mathrm{T}}\varepsilon_i - 2\sum_{i=1}^{N-1} \varepsilon_i^{\mathrm{T}} P\bar{L}(t)\varepsilon_i + 2\sum_{i=1}^{N-1} \|\varepsilon_i\| \, \|P\| \left(\sum_{k=1}^{n} |F_{ik}(t)|\right) + \tilde{M}(t)\dot{\hat{M}}(t). \tag{14}$$

By using Assumptions 1 and 2 and transformation (6), we have

$$\begin{aligned}\sum_{k=1}^{n} |F_{ik}| &\leq \sum_{k=1}^{n} \rho_i \sum_{j=1}^{k} \frac{|e_{ij}|}{L^k(t)} = \rho_i \sum_{j=1}^{n} |e_{ij}| \sum_{k=j}^{n} \frac{1}{L^k(t)} \\ &< 2\rho_i \sum_{j=1}^{n} \frac{|e_{ij}|}{L^j(t)} = 2\rho_i \sum_{j=1}^{n} |\varepsilon_{ij}| \leq 2\rho\sqrt{n}\, \|\varepsilon_i\| \,.\end{aligned} \tag{15}$$

Choosing $L(t) = L_1 + \alpha\hat{M}(t)$ with constant $L_1 > 2$ and $\alpha = 4\sqrt{n}\,\|P\|$, we have

$$\frac{L(t)}{L(t)-1} < 2, \quad \text{for any } L(t) > 2.$$

The update law of $\hat{M}(t)$ is taken as

$$\dot{\hat{M}}(t) = 4\sqrt{n}\,\|P\| \sum_{i=1}^{N-1} \varepsilon_i^{\mathrm{T}}\varepsilon_i, \qquad \hat{M}(0) > 0. \tag{16}$$

By calculating and the condition of Theorem 1, we obtain

$$\begin{aligned}&\sum_{i=1}^{N-1} \varepsilon_i^{\mathrm{T}}\varepsilon_i = \sum_{i=1}^{N-1}\sum_{k=1}^{n} \frac{e_{ik}^2}{L^{2k}(t)} \leq \frac{1}{4}\sum_{i=1}^{N-1} e_i^{\mathrm{T}} e_i = \frac{1}{4}\|e\|^2 \leq D \quad \text{and} \\ &\frac{\mathrm{d}\ln L^k(t)}{\mathrm{d}t} = \frac{k\alpha\dot{\hat{M}}(y)}{L(t)} < \frac{n}{2}\alpha^2\varepsilon^{\mathrm{T}}\varepsilon \leq \frac{n}{2}\alpha^2 D,\end{aligned} \tag{17}$$

where D is an unknown constant.

By using the inequality $2X^{\mathrm{T}}Y \leq X^{\mathrm{T}}X + Y^{\mathrm{T}}Y$, $X, Y \in R^n$, (15)–(17), we can rewrite the (14) as

$$\begin{aligned}\frac{\mathrm{d}V(t)}{\mathrm{d}t} &\leq -\sum_{i=1}^{N-1}\left(L(t) - 4\sqrt{n}\rho\,\|P\|\right)\varepsilon_i^{\mathrm{T}}\varepsilon_i + \sum_{i=1}^{N-1}\varepsilon_i^{\mathrm{T}}P^2\varepsilon_i + \sum_{i=1}^{N-1}\varepsilon_i^{\mathrm{T}}\bar{L}^2(t)\varepsilon_i + \tilde{M}(t)\dot{\hat{M}}(t). \\ &\leq -\sum_{i=1}^{N-1}\left(L_1 + \alpha\hat{M}(t) - 4\sqrt{n}\rho\,\|P\| - \lambda_{\max}\left(P^2\right) - \frac{n^2}{4}\alpha^4 D^2\right)\varepsilon_i^{\mathrm{T}}\varepsilon_i + \tilde{M}(t)\dot{\hat{M}}(t) \\ &= -\left(L_1 - \lambda_{\max}\left(P^2\right)\right)\sum_{i=1}^{N-1}\varepsilon_i^{\mathrm{T}}\varepsilon_i + \tilde{M}(t)\left(-4\sqrt{n}\,\|P\|\sum_{i=1}^{N-1}\varepsilon_i^{\mathrm{T}}\varepsilon_i + \dot{\hat{M}}(t)\right) \\ &\leq -\left(L_1 - \lambda_{\max}\left(P^2\right)\right)\sum_{i=1}^{N-1}\varepsilon_i^{\mathrm{T}}\varepsilon_i.\end{aligned}$$

We obtain the time derivative of $V(t)$ as follows

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} \leq -W\left(\varepsilon, \tilde{M}\right) \leq 0, \tag{18}$$

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where $W\left(\varepsilon,\tilde{M}\right)=\left(L_1-\lambda_{\max}\left(P^2\right)\right)\sum_{i=1}^{N-1}\varepsilon_i^{\mathrm{T}}\varepsilon_i$ and $L_1>\max\left\{2,\lambda_{\max}\left(P^2\right)\right\}$.

Since $\frac{\mathrm{d}V(t)}{\mathrm{d}t}\le 0$, $V(t)$ is decreasing. Thus, in view of the inequalities $\lambda_{\min}(P)\sum_{i=1}^{N-1}\varepsilon_i^{\mathrm{T}}\varepsilon_i+\frac{\tilde{M}^2}{2}\le V(t)\le\lambda_{\max}(P)\sum_{i=1}^{N-1}\varepsilon_i^{\mathrm{T}}\varepsilon_i+\frac{\tilde{M}^2}{2}$, we conclude that the solutions of systems (9) at $\varepsilon(t)=0$, $\tilde{M}(t)=0$ are uniformly stable on the any bounded region $B_D=\left\{e:e^{\mathrm{T}}e\le D, D>0\right\}$, Moreover, the solutions of systems (9) $\varepsilon(t)$, $\tilde{M}(t)$ are uniformly bounded on the region B_D, that is $\left\|\left(\varepsilon^{\mathrm{T}}(t),\tilde{M}(t)\right)^{\mathrm{T}}\right\|\le B$, $\forall t\ge 0$. Since $V(t)$ is not increasing and bounded from below by zero, we have

$$\lim_{t\to\infty}\int_{t_0}^{t}W\left(\varepsilon(\tau),\tilde{M}(\tau)\right)\mathrm{d}\tau\le V(t)-V_\infty,$$

which means that $\int_{t_0}^{\infty}W\left(\varepsilon(\tau),\tilde{M}(\tau)\right)\mathrm{d}\tau$ exists and is finite. Moreover,

$$W\left(\varepsilon,\tilde{M}\right),\ \frac{\mathrm{d}W\left(\varepsilon,\tilde{M}\right)}{\mathrm{d}t}\in L_\infty.$$

By Barbalat lemma [15], we conclude that

$$\begin{aligned}\lim_{t\to\infty}W\left(\varepsilon,\tilde{M}(t)\right)&=\lim_{t\to\infty}\left(L_1-\lambda_{\max}\left(P^2\right)\right)\sum_{i=1}^{N-1}\varepsilon_i^{\mathrm{T}}(t)\varepsilon_i(t)\\&=\left(L_1-\lambda_{\max}\left(P^2\right)\right)\lim_{t\to\infty}\sum_{i=1}^{N-1}\|\varepsilon_i(t)\|^2=0.\end{aligned}$$

It follows that $\lim_{t\to\infty}\|\varepsilon_i(t)\|=0$. □

By applying the Theorem 1, we can easily obtain the following result.

Corollary 1. *Let Assumptions 1 and 2 hold. Then, for any unknown bounded region $B_D=\left\{e:e^{\mathrm{T}}e\le D, D>0\right\}$, D is an arbitrary unknown constant and $e=\left(e_1\ \cdots\ e_{N-1}\right)^{\mathrm{T}}$, there exist the scalar robust adaptive controllers given in (10) such that solutions of systems (1) synchronize with each other in the sense that (2).*

Proof. By Theorem 1, we have obtain $\lim_{t\to\infty}\|\varepsilon_i(t)\|=0$, $i=1,2,\ldots,N-1$. It follows that $\lim_{t\to\infty}\|e_i\|=0$, $i=1,2,\ldots,N-1$ hold along with $\varepsilon_{ik}=\frac{e_{ik}}{L^k(t)}$, $i=1,2,\ldots,N-1$, $k=1,2,\ldots,n$.

For any $i,j=1,\ldots,N$, since

$$\left\|x_i-x_j\right\|=\left\|(x_i-x_{i+1})+(x_{i+1}-x_{i+2})+\cdots\left(x_{j-1}-x_j\right)\right\|\le\sum_{k=i}^{j-1}\|e_k\|,$$

along with $\lim_{t\to\infty}\|e_k\|=0$, $k=1,2,\ldots,N-1$, we have $\lim_{t\to\infty}\left\|x_i(t)-x_j(t)\right\|=0$, which means that the solutions of (1) satisfy (2).

The proof is thus completed. □

4. Simulation

Consider the Rössler system [16]

$$\begin{aligned}&\dot{x}=-y-z,\\&\dot{y}=x+ay,\\&\dot{z}=b+z\left(x-c\right).\end{aligned}\tag{19}$$

It has been proved that (19) presents chaos when $a=b=0.2$, $c=5.7$.

Introducing a linear transformation $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0&1&0\\1&0&0\\0&0&-1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}$.

By using Rössler system (19) as the nodes of the dynamical network and choosing nearest-neighbor coupling functions

$$f_{i1}=ax_{i1}+\tau x_{i1}+\sum_{k=i}^{i+2}c_{k1}(t)\tanh(x_{k1}),\qquad f_{i2}=-x_{i1},\qquad f_{i3}=-b+x_{i3}(x_{i2}-c),\quad i=1,2,\ldots,N-2,$$

and

$$\begin{aligned}&f_{i,1}=ax_{i1}+\tau x_{i1}+c_{i,1}(t)\tanh\left(x_{i-1,1}\right)+2c_{i,1}(t)\tanh\left(x_{i,1}\right),\\&f_{i2}=-x_{i1},\qquad f_{i3}=-b+x_{i3}\left(x_{i2}-c\right),\quad i=N-1,N\end{aligned}\tag{20}$$

where $c_{k1}(t)=\cos t$, $k=1,2,\ldots,N$, $\tau=1+\sin^2 t$,

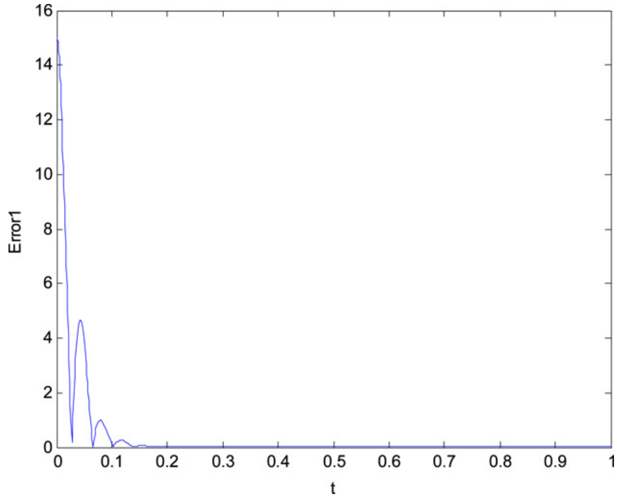

Fig. 1. The synchronization error-states $Error1 = \sqrt{\sum_{i=1}^{499} e_{i1}^2}$.

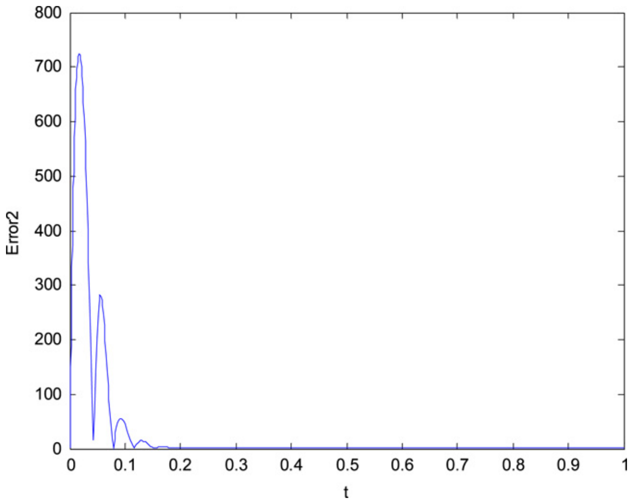

Fig. 2. The synchronization sum error-states $Error2 = \sqrt{\sum_{i=1}^{499} e_{i2}^2}$.

we obtain the following dynamical network with N identical Rössler systems

$$\begin{aligned} \dot{x}_{i1} &= x_{i2} + ax_{i1} + \tau x_{i1} + \sum_{k=i}^{i+2} c_{k1}(t)\tanh(x_{k1}), \\ \dot{x}_{i2} &= x_{i3} - x_{i1}, \\ \dot{x}_{i3} &= u_i - b + x_{i3}(x_{i2} - c), \quad i = 1, 2, \ldots, N-2, \end{aligned}$$

and

$$\begin{aligned} \dot{x}_{i1} &= x_{i2} + ax_{i1} + \tau x_{i1} + c_{i1}(t)\tanh\left(x_{i-1,1}\right) + 2c_{i1}(t)\tanh(x_{i1}) \\ \dot{x}_{i2} &= x_{i3} - x_{i1}, \\ \dot{x}_{i3} &= u_i - b + x_{i3}(x_{i2} - c), \quad i = N-1, N. \end{aligned} \tag{21}$$

It is clear that both the Assumption 1 and the Assumption 2 are satisfied, which follows that (20) and the solution of Rössler system (19) is bounded.

In simulation, we choose the network size, $N = 500$, and

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -5 \end{bmatrix}, \qquad P = \begin{bmatrix} 9.6667 & -0.5000 & -1.8333 \\ -0.5000 & 1.8333 & -0.5000 \\ -1.8333 & -0.5000 & 0.6667 \end{bmatrix}, \qquad L_1 = 200.$$

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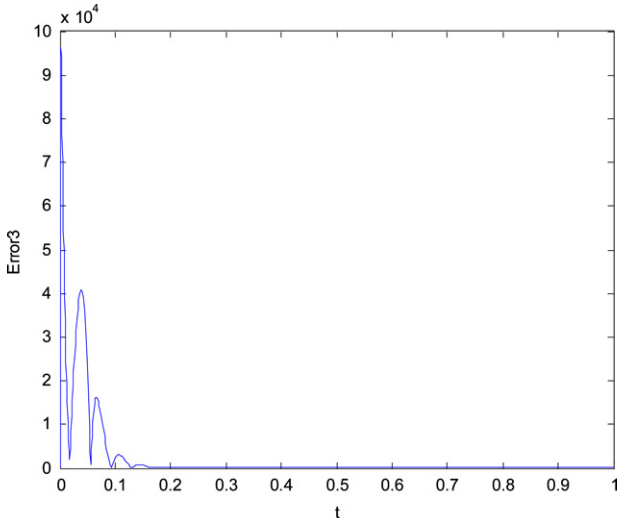

Fig. 3. The synchronization error-states $Error3 = \sqrt{\sum_{i=1}^{499} e_{i3}^2}$.

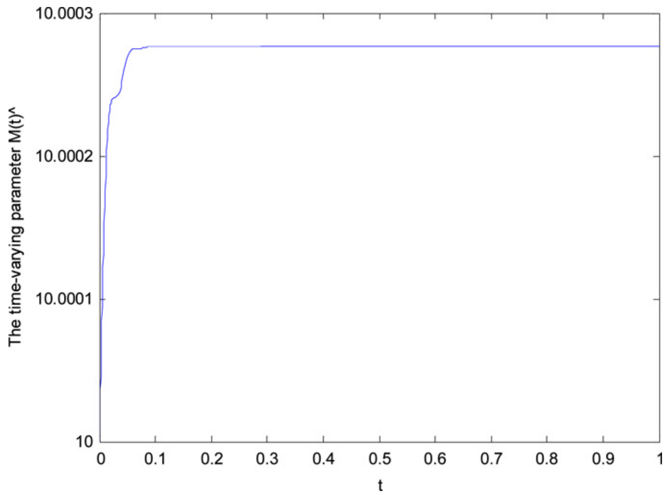

Fig. 4. The time-varying parameter $\hat{M}(t)$.

We choose scalar controller $u_1 = 0$, other scalar state feedback controllers $u_i, i = 2, \ldots, 500$ and update law are constructed according to the formula (10) and (11) of the Theorem 1.

We use the following three error quantities

$$Error1 = \sqrt{\sum_{i=1}^{499} e_{i1}^2} = \sqrt{\sum_{i=1}^{499} \left(x_{i1}(t) - x_{i+1,1}(t)\right)^2}, \qquad Error2 = \sqrt{\sum_{i=1}^{499} e_{i2}^2} = \sqrt{\sum_{i=1}^{499} \left(x_{i2}(t) - x_{i+1,2}(t)\right)^2},$$

$$Error3 = \sqrt{\sum_{i=1}^{499} e_{i3}^2} = \sqrt{\sum_{i=1}^{499} \left(x_{i3}(t) - x_{i+1,3}(t)\right)^2}$$

to illustrate the behaviors of synchronization in network (21) with the scalar robust adaptive controllers designed according to Theorem 1.

The results of the simulation are as Figs. 1–5.

Figs. 1–3 clearly show that when adding some scalar controllers constructed as Theorem 1 on the dynamical network (21), the error states between the dynamical network approach zeros as the time approaches infinitely. It means that the states of the dynamical network robust asymptotically synchronize with each other. Fig. 4 shows that time-varying parameter $\hat{M}(t)$ changes in robust adaptive controllers with the states of dynamical network (21) synchronizing in the sense of that (2). From Fig. 4, we can see that $\hat{M}(t) \to$ constant, $t \to \infty$, which agrees with update law (11) $\dot{\hat{M}}(t) \to 0, t \to \infty$. In

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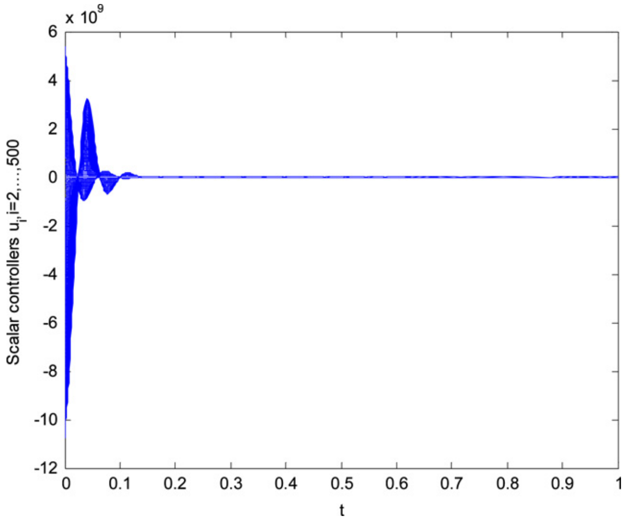

Fig. 5. Scalar controllers $u_i(t)$, $i = 2, \dots, 500$.

Fig. 5, we show that how much the energy of the control inputs $u_i(t)$, $i = 2, \dots, 500$ we have to provide in order to grant synchronization for dynamical network (21).

5. Conclusion

The high gain feedback control theory is extended to solving the issue of robust adaptive synchronization of dynamical network in this paper. When the non-linear coupling functions are unknown but with unknown bounded, we introduce the time-varying gain parameter in the designing of the scalar controllers which are able to make the states of uncertain coupled dynamical networks robust adaptive asymptotically synchronize with each other. The obtained theoretical result has been validated by the numerical simulation.

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